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An interior transmission eigenvalue problem

Diplomarbeit

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Preface

Before we start with the mathematics I want to very briefly thank my parents and grandparents for providing me with financial support during my studies.

Of course I also owe thanks to Professor Kirsch, who gave me the topic and always took his time for me whenever I had a problem.

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Chapter 1

Introduction

What we are generally concerned with is the scattering of an incoming wave at an inhomogeneous medium. Interior transmission eigenvalues appear in that context and the question whether they exist is of importance to determine whether certain methods of reconstructing the scattering object are guaranteed to succeed [4]. As mentioned in [3] there are also concepts that use knowledge about the distribution of interior transmission eigenvalues to directly gather information about the scattering object.

Our goal is to prove that under certain conditions interior transmission eigenvalues exist, but only as isolated points. Points of reference that are of utmost importance to this end are the research papers [3] and [9]. We will either mimic their proofs or work them out in greater detail in the central chapters 3 and 4 where we tackle the necessary details for achieving our aforementioned goal. Before that we will be discussing the basics needed for this in chapter 2. Chapter 5 will then cover some loose ends and provide some exemplary analysis in a more specific environment.

Definitions 1.1 and 1.2 set the stage and the quantities defined in them will have the properties attributed to them there throughout the entire thesis unless stated otherwise.

Definition 1.1. *Let $D \subseteq \mathbb{R}^2$ be a bounded domain (i.e. connected and open) of class C^2 with connected complement. Furthermore let $p, q \in L^\infty(D)$ such that there exists a constant $q_0 > 0$ with $q(x) \geq p(x) + q_0 \geq q_0$ for almost all $x \in D$.*

Keeping in mind the affinity to the scattering problem it makes sense to visualize the variables in this definition as physical parameters.

p describes the index of refraction for some kind of background medium and q the index of refraction for the actual scattering object. q_0 can be imagined as a minimum requirement for contrast as $q - p$ is bounded away from zero by it.

D contains the scatterer. It is defined to be a subset of $\mathbb{R}^2$ which means we are restricting ourselves to the two-dimensional problem. I am not aware that examining the three-dimensional problem would pose serious additional difficulties, however.

The smoothness condition 'class C^2' means that the boundary can locally be written as a twice continuously differentiable function after an adequate transformation of the coordinate system. This is needed to use some embedding theorems concerning Sobolev spaces that will prove to be very helpful, but could most likely be slightly weakened without destroying the validity of the theorems. While the boundedness of D seems like a strong restriction from a theoretical point of view, one has to keep in mind that boundedness of the scatterer is not much of a restriction when the scatterer is an actual physical object, like a patient in a medical imaging device or a test specimen subjected to non-destructive testing techniques.

Definition 1.2. $\lambda > 0$ *is called* ***interior transmission eigenvalue*** *if there exist nontrivial* $u, w \in L^2(D)$ *such that*

1. $u - w \in H_0^2(D)$
2. $\Delta u + \lambda(1+q)u = 0$ *in* D
3. $\Delta w + \lambda(1+p)w = 0$ *in* D
4. $u = w$ *and* $\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu}$ *on the boundary* ∂D *of* D

whereas 2. *and* 3. *have to be understood in the weak sense, which means that*

$$0 = \iint_D u \cdot (\Delta + \lambda(1+q))\Psi \, dx \quad \forall \ \Psi \in H_0^2(D) \text{ and}$$

$$0 = \iint_D w \cdot (\Delta + \lambda(1+p))\Psi \, dx \quad \forall \ \Psi \in H_0^2(D) \text{ respectively.}$$

All of the aforementioned functions and function spaces are required to be real and will be properly defined in due time.

Chapter 2

Preliminaries

This chapter is devoted to the foundation for our proofs and calculations and includes some general definitions and theorems, mostly from functional analysis. The intention here is not to construct a complete analytical framework from scratch, but to recall some definitions and well-known facts so we have them readily available later with suitable notation and level of detail for our purposes. Furthermore stating theorems in their most general form is not the goal either. In favor of added consistency we will for example state most theorems for Banach- and Hilbert spaces, even if the theorems were also true in less demanding spaces.

For the proofs of the essential theorems the reader will mostly be referred to one of the textbooks I used while writing. Whenever we prove something by ourselves in this chapter, it is likely not to stress the importance of that particular statement, but rather because the statement is uncommon in that exact formulation. There are some exceptions to this rule, though.

As far as our definitions are concerned, these are intended to be as complete as possible without taking up too much space. I tried to avoid situations where the reader might be unsure what a given object is, but defining everything, no matter how basic, was obviously not the way to go either. Hopefully a satisfactory balance between completeness and clarity has been found.

2.1 From norms to Hilbert spaces

The source for all of the definitions in this section is [12], they are however not taken word for word. We take the knowledge what a vector space, eigenvalues and eigenvectors are for granted and proceed to recall the definition of norms and inner products.

Definition 2.1. *Let X be a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.*
A mapping $\|.\| \colon X \to \mathbb{R}$ is called a norm if it satisfies

$$\|x\| \geq 0, \ \ \|x\| = 0 \iff x = 0$$

$$\|\alpha x\| = |\alpha| \|x\| \qquad \textit{Positive homogeneity}$$

$$\|x + y\| \leq \|x\| + \|y\| \qquad \textit{Triangle inequality}$$

for all $\alpha \in \mathbb{F}$ *and all* $x, y \in X$*.*
A vector space with a norm is called a normed vector space.

Definition 2.2. *Let* X *be a vector space over* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$*.*
A mapping $\langle .,. \rangle : X \times X \to \mathbb{F}$ *is called an inner product if it satisfies*

$$\langle x_1 + \alpha x_2, y \rangle = \langle x_1, y \rangle + \alpha \langle x_2, y \rangle \qquad \textit{Linearity}$$
$$\langle x, y \rangle = \overline{\langle y, x \rangle} \qquad \textit{(Conjugate) Symmetry}$$
$$\langle x, x \rangle \geq 0, \ \langle x, x \rangle = 0 \iff x = 0 \qquad \textit{Positive definiteness}$$

for all $\alpha \in \mathbb{F}$ *and all* $x, y, x_1, x_2 \in X$*.*
A vector space with an inner product is called an inner product space.

Remark 2.3. *In Definition 2.2 the use of the relation '* $\geq$ *' implicitly defines that the numbers on both sides of the relation must be on the real axis if they are otherwise possible to be anywhere in* $\mathbb{C}$*. This procedure applies with '* $\geq$ *' as well as with '* $>$ *', '* $<$ *' and '* $\leq$ *' everywhere in this thesis, in particular also in Definition 1.2. The complex conjugate of a real number is defined to be the number itself. This holds still true, if the field that we are currently working in is not even* $\mathbb{C}$ *to begin with, which allows us to simultaneously treat the real and the complex case for some problems.*

Recall that if we have an inner product we can define a norm through

$$\|x\| := \sqrt{\langle x, x \rangle}$$

(see [12, Lemma 6.20] for a short proof). If not specifically stated otherwise we always think of this norm when talking about the norm of a vector in an inner product space.

A very well-known device that is essential to the proof that $\sqrt{\langle x, x \rangle}$ is a norm, but also is interesting on it's own is the Cauchy-Schwarz inequality. We cite it from [12, Lemma 6.20].

Lemma 2.4. *Let* X *be an inner product space.*
Then the Cauchy-Schwarz inequality

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle = \|x\|^2 \|y\|^2$$

is valid for all x *and* y *in* X*.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned} 0 &\leq \langle x - \frac{\langle x, y \rangle}{\langle y, y \rangle} y, x - \frac{\langle x, y \rangle}{\langle y, y \rangle} y \rangle \\ &= \langle x, x \rangle - \frac{\langle x, y \rangle}{\langle y, y \rangle} \langle y, x \rangle - \frac{\overline{\langle x, y \rangle}}{\langle y, y \rangle} \langle x, y \rangle + \frac{\langle x, y \rangle \overline{\langle x, y \rangle}}{\langle y, y \rangle^2} \langle y, y \rangle \\ &= \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} \end{aligned}$$

from where the Cauchy-Schwarz inequality follows by adding $\frac{|\langle x,y\rangle|^2}{\langle y,y\rangle}$ on both sides and then multiplying with $\langle y, y\rangle$. □

Definition 2.5. *Let X be a normed vector space.*
When we say a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to $x \in X$, we mean norm convergence, i.e.

$$\|x_n - x\| \to 0$$

and write $x_n \to x$. When we say $(x_n)_{n\in\mathbb{N}}$ is bounded, then we mean that $(\|x_n\|)_{n\in\mathbb{N}}$ is bounded as a sequence in $\mathbb{R}$. We call $(x_n)_{n\in\mathbb{N}}$ a Cauchy sequence, if it has the property

$$\forall\ \epsilon > 0\ \exists\ N \in \mathbb{N} : \|x_n - x_m\| < \epsilon \quad \forall\ n, m > N$$

Last but not least we define the two types of vector spaces that are most important for our proofs and calculations.

Definition 2.6. *A normed vector space X is called complete, if every Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to some $x \in X$.*
A complete normed vector space is called a Banach space.

Definition 2.7. *A complete inner product space is called a Hilbert space.*

2.2 On operators

We will begin this section by giving some definitions of properties of operators using the same notation as [11], which is also the source of the definitions, albeit again with slightly different formulations. After that we will recall convenient equivalent conditions for some of the defined properties and a theorem that guarantees the existence of a square root of an operator under certain circumstances. The results are well-known and thus heavily rely on citations for their proofs, leaving only minor details left to prove ourselves. Since in our case the domain will always be the whole vector space on which the operator is defined, we do not need to develop a general approach on that aspect.

Definition 2.8. *A mapping $Q\colon X \to Y$ between two vector spaces X and Y over the same field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ is called a linear operator if it satisfies*

$$Q(\alpha x_1 + x_2) = \alpha Q(x_1) + Q(x_2)$$

for all $\alpha \in \mathbb{F}$ and all $x_1, x_2 \in X$. Note that the addition and scalar multiplication on the left hand side and the right hand side of the equation are not necessarily the same.

Following convention we will from now on we drop the brackets and write Qx instead of $Q(x)$. Furthermore Definition 2.8 implicitly states that throughout the section any two vector spaces that are connected through a linear operator are over the same field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. If we have only one vector space, then we also denote it's field by $\mathbb{F}$.

Definition 2.9. *A linear operator $Q\colon X \to Y$ between two normed vector spaces X and Y is called bounded if there exists $C > 0$ such that*

$$\|Qx\|_Y \leq C\|x\|_X \quad \forall\, x \in X \tag{2.1}$$

In this case

$$\|Q\| := \inf\{C > 0 \mid (2.1) \textit{ holds true}\}$$

is called the operator norm of Q.

Lemma 2.10. *Let $Q_1\colon X \to Y$ and $Q_2\colon X \to Y$ be bounded linear operators and $\alpha \in \mathbb{F}$. Then $\|.\|$ as defined in Definition 2.9 satisfies*

$$\|Q_1 + Q_2\| \leq \|Q_1\| + \|Q_2\| \qquad\qquad \|\alpha Q_1\| = |\alpha|\|Q_1\|$$

This lemma is proven in e.g. [11, Section 4.4] and also should give a good idea why we call the operator norm 'norm'. Imagining operators as elements of normed vector spaces is however not necessary in our context.
[11, Section 5.13] motivates the following definition of self-adjointness, which is sufficient for our purposes.

Definition 2.11. *A linear operator $Q\colon H \to H$ from a Hilbert space H onto itself is called self-adjoint if it satisfies*

$$\langle Qh_1, h_2\rangle = \langle h_1, Qh_2\rangle$$

for all $h_1, h_2 \in H$.

What we will need at one point is to take a square root of a self-adjoint bounded linear operator Q, i.e. we need to find an operator $Q^{\frac{1}{2}}$ such that $Q^{\frac{1}{2}}Q^{\frac{1}{2}} = Q$. However self-adjointness is not enough to ensure the existence of such an operator as one can visualize for example by taking $\mathbb{R}$ as a vector space over itself with the multiplication as inner product. Linear operators are in this case simply multiplications with a given real number and are automatically bounded and self-adjoint. If we choose Q to be multiplication by a negative number, we can not find a square root of Q in the set of linear operators. Hence we need to assume one more property, namely nonnegativeness.

Definition 2.12. *A self-adjoint bounded linear operator $Q\colon H \to H$ from a Hilbert space H onto itself is called nonnegative if it satisfies*

$$\langle Qh, h\rangle \geq 0$$

for all $h \in H$.

Definition 2.13. *We denote the set of all nonnegative self-adjoint bounded linear operators from a Hilbert space H onto itself by $\mathcal{B}^+[H]$.*

Now we are able to cite the following lemma from [11, Theorem 5.85], where one can also find a very detailed and nice proof.

Lemma 2.14. *For every operator $Q \in \mathcal{B}^+[H]$ there exists a unique $Q^{\frac{1}{2}} \in \mathcal{B}^+[H]$ with the property that $Q^{\frac{1}{2}}Q^{\frac{1}{2}} = Q$.*

We call this $Q^{\frac{1}{2}}$ the square root of Q. For our application of this lemma we also need to ensure that the square root is compact if the original operator is compact. Our definition of compactness is based on [11, Theorem 4.52], where it is shown that this definition is equivalent to a definition with sets instead of sequences.

Definition 2.15. *A linear operator $Q\colon X \to Y$ between two Banach spaces X and Y is called compact if for every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$ there exists a convergent subsequence in $(Qx_n)_{n\in\mathbb{N}} \subset Y$.*

Definition 2.16. *Let X be a normed vector space. The set of all bounded linear operators from X to $\mathbb{F}$ is called the dual space of X and is denoted by X^*. Here we identify the field $\mathbb{F}$ with the vector space $\mathbb{F}^1$.*

Definition 2.17. *We say a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges weakly to $x \in X$ if $f(x_n) \to f(x)$ in $\mathbb{F}$ for all $f \in X^*$. In this case we write $x_n \xrightarrow{w} x$.*

Three well-known facts (see e.g. [11, Problem 4.67]) are that weak convergence implies boundedness, that norm convergence implies weak convergence and that weak limits are unique.

Lemma 2.18. *Let X be a Banach space and $x, x_1^\infty, x_2^\infty \in X$.*
If $(x_n)_{n\in\mathbb{N}} \subset X$ is such that $x_n \xrightarrow{w} x$, then $(x_n)_{n\in\mathbb{N}}$ is bounded. If $(x_n)_{n\in\mathbb{N}} \subset X$ is such that $x_n \to x$, then $x_n \xrightarrow{w} x$. If $(x_n)_{n\in\mathbb{N}} \subset X$ is such that $x_n \xrightarrow{w} x_1^\infty$ and $x_n \xrightarrow{w} x_2^\infty$, then $x_1^\infty = x_2^\infty$.

An equivalent condition for compactness on Hilbert spaces can be found in [11, Problem 5.41] where it is an exercise with hints. For better readability we will transform the hints into a rigorous proof without deviating from the idea of [11].

Lemma 2.19. *A linear operator $Q\colon X \to Y$ between two Hilbert spaces X and Y is compact if and only if $Qx_n \to 0$ in Y for all $(x_n)_{n\in\mathbb{N}} \subset X$ that converge weakly to zero in X.*

Proof. First let Q be compact and choose $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \overset{w}{\to} 0$ in X. Observe that $(x_n)_{n\in\mathbb{N}}$ is bounded by Lemma 2.18.

We show that $Qx_n \overset{w}{\to} 0$ is true in Y. To that end let $f \in Y^*$. Due to the linearity of both Q and f, $fQ\colon X \to \mathbb{F}$ is also linear:

$$\begin{aligned} fQ(\alpha x_1 + x_2) &= f(Q(\alpha x_1 + x_2)) = f(\alpha Q(x_1) + Q(x_2)) \\ &= \alpha f(Q(x_1)) + f(Q(x_2)) = \alpha fQ(x_1) + fQ(x_2) \end{aligned}$$

for all $\alpha \in \mathbb{F}$ and all $x_1, x_2 \in X$. Hence $fQ \in X^*$. Because $x_n \overset{w}{\to} 0$ in X we see that for any $f \in Y^*$ we have $f(Qx_n) = fQ(x_n) \to 0$ in $\mathbb{F}$ which proves that $Qx_n \overset{w}{\to} 0$ in Y.

Assume that $(Qx_n)_{n\in\mathbb{N}}$ does not converge to zero. Then there must exist a subsequence $(Qx_{n_k})_{k\in\mathbb{N}}$ that is bounded away from zero in norm, i.e. there exists $\epsilon > 0$ such that

$$\|Qx_{n_k}\|_Y \geq \epsilon \tag{2.2}$$

for all $k \in \mathbb{N}$.

Due to the boundedness of $(x_n)_{n\in\mathbb{N}}$ and the compactness of Q, $(Qx_{n_k})_{k\in\mathbb{N}}$ must have a convergent subsequence $(Qx_{n_{k_j}})_{j\in\mathbb{N}} \subset Y$. Let $y \in Y$ denote the limit of this subsequence. By (2.2) we see $\|y\|_Y \geq \epsilon$ and thus $y \neq 0$. As norm convergence implies weak convergence (Lemma 2.18) we have $Qx_{n_{k_j}} \overset{w}{\to} y$ in Y. This is a contradiction to the fact that we have already $Qx_n \overset{w}{\to} 0$ and thus also $Qx_{n_{k_j}} \overset{w}{\to} 0$ in Y because weak limits are unique (Lemma 2.18 again). Hence our assumption is wrong and $Qx_n \to 0$ in Y.

Now let Q have the property that $Qx_n \to 0$ for any $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \overset{w}{\to} 0$ in X.

Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a bounded sequence. We know that every bounded sequence in a Hilbert space has a weakly convergent subsequence (see e.g. [11, Lemma 5.69] for a proof). Hence there exists $x \in X$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \overset{w}{\to} x$ in X.

$x_{n_k} - x \overset{w}{\to} 0$ in X can easily be seen from the definition of weak convergence. Let $f \in X^*$, then $f(x_{n_k} - x) = f(x_{n_k}) - f(x) \to 0$ in $\mathbb{F}$ if we already know that $f(x_{n_k}) \to f(x)$.

Applying the special property of Q onto the sequence $(x_{n_k} - x)_{k\in\mathbb{N}}$ we get that $Qx_{n_k} - Qx = Q(x_{n_k} - x) \to 0$ and thus $Qx_{n_k} \to Qx$. Or in other words we have found a convergent subsequence of $(Qx_n)_{n\in\mathbb{N}}$. According to Definition 2.15 this means that Q is compact. □

With Lemma 2.19 proven, we can now prove the lemma that guarantees the compactness of our square root of a compact operator. It can be found in [11, Problem 5.62], from where we will transform the hints into a rigorous proof once more.

Lemma 2.20. *Let* $Q \in \mathcal{B}^+[H]$*. If* Q *is compact, then it's square root* $Q^{\frac{1}{2}} \in \mathcal{B}^+[H]$ *from Lemma 2.14 is also compact.*

Proof. Due to Lemma 2.19 it is sufficient to show that $Q^{\frac{1}{2}}x_n \to 0$ for all $(x_n)_{n\in\mathbb{N}} \subset H$ that converge weakly to zero. On that account let $(x_n)_{n\in\mathbb{N}} \subset H$ be such that $x_n \overset{w}{\to} 0$.

$$\|Q^{\frac{1}{2}}x_n\|^2 = \langle Q^{\frac{1}{2}}x_n, Q^{\frac{1}{2}}x_n\rangle = \langle Qx_n, x_n\rangle \leq \|Qx_n\|\|x_n\| \tag{2.3}$$

is true due to the self-adjointness of $Q^{\frac{1}{2}}$ and the Cauchy-Schwarz inequality (Lemma 2.4). $\|Qx_n\| \to 0$ due to Lemma 2.19 because Q is compact. $(\|x_n\|)_{n\in\mathbb{N}}$ is bounded due to Lemma 2.18 because $(x_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence.
Hence the right hand side of (2.3) converges to zero and thus $Q^{\frac{1}{2}}x_n \to 0$.
Now $Q^{\frac{1}{2}}$ is compact by Lemma 2.19. □

Lemma 2.21. *Let* $Q_1 \colon X \to Y$ *and* $Q_2 \colon X \to Y$ *be compact linear operators between two Banach spaces* X *and* Y*. Then* $\alpha Q_1 + Q_2$ *is also a compact linear operator for all* $\alpha \in \mathbb{F}$*.*

We refer to [11, Theorem 4.53] but note that the proof could also be done quickly as an exercise.

2.2.1 Spectral theory

Of utmost importance for chapter 3 is a spectral theorem for compact operators. Let us cite it from [14, Theorem VI.2.5] first and then discuss what the new notation means and how it fits into our setting afterwards. Note that original version of this theorem encompasses more than we need and we do not cite it wholesale, and that we refer to [14] for the proof.

Theorem 2.22. *Let* X *be a Banach space over* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ *and* $Q \colon X \to X$ *be a compact linear operator.*

a) *If* X *is of infinite dimension, then* $0 \in \sigma(Q)$*.*

b) *The (possibly empty) set* $\sigma(Q)\backslash\{0\}$ *is countable.*

c) *Every* $\lambda \in \sigma(Q)\backslash\{0\}$ *is an eigenvalue of* Q*.*

d) $\sigma(Q)$ *does not possess an accumulation point other than* 0*.*

Definition 2.23. *The resolvent set $\rho(Q)$ of Q is the set of all $\lambda \in \mathbb{F}$ such that $\lambda I - Q$ has a bounded inverse. Here I denotes the identity on X.*
$\sigma(Q) := \mathbb{F} \backslash \rho(Q)$ is called the spectrum of Q.

In general the spectrum of a linear operator between two Banach spaces consists of three disjoint sets called the point spectrum, the continuous spectrum and the residual spectrum whereas the set of eigenvalues is the point spectrum [12, Definition 8.39]. However we do not need to delve deeply into that because the theorem states that except possibly 0, any λ in the spectrum is an eigenvalue, i.e. there exists $x \in X$, $x \neq 0$, such that $Qx = \lambda x$.

Furthermore there can not be eigenvalues of Q in $\rho(Q)$ as we see from the definition of $\rho(Q)$ that $\lambda I - Q$ must be injective for any $\lambda \in \rho(Q)$. And because $x = 0$ already solves the equation $\lambda I x - Qx = 0$, this means there can be no other solution.

If additionally to compactness we have self-adjointness, we can conclude even more. We cite a slimmed down (dropping the case $\mathbb{F} = \mathbb{C}$ and a decomposition of H) version of the spectral theorem for compact self-adjoint operators from [14, Theorem VI.3.2], where a proof can also be found.

Theorem 2.24. *Let $Q\colon H \to H$ be a compact self-adjoint linear operator from a real Hilbert space H onto itself. Then there exists an (possibly finite) orthonormal system $\{e_1, e_2, \cdots\}$ and a zero sequence $(\lambda_1, \lambda_2, \cdots) \subset \mathbb{R}\backslash\{0\}$ such that*

$$Qx = \sum_k \lambda_k \langle x, e_k \rangle e_k \tag{2.4}$$

The λ_k are the nonzero eigenvalues and e_k is an eigenvector to λ_k. Furthermore we have $\|Q\| = \sup_k |\lambda_k|$.

'Orthonormal system' means that each e_k has norm one and the vectors are pairwise orthogonal, i.e. $\langle e_k, e_j \rangle = 0$ for $j \neq k$. This theorem is a massively useful tool to handle compact self-adjoint operators, the only downside is that the requirements are rather strict. Observe furthermore that $\{e_1, e_2, \cdots\}$ is not empty for $Q \neq 0$ due to (2.4).

The decomposition of the space H within the original theorem that we did not cite above will be used by us in the form of the following lemma which can be seen from the proof of [14, Theorem VI.3.2] as well.

Lemma 2.25. *In the setting of Theorem 2.24 any $x \in H$ can be written as*

$$x = y + \sum_k \langle x, e_k \rangle e_k$$

for some y in the kernel of Q. Furthermore y is in the orthogonal complement of the closure of span$\{e_1, e_2, \cdots\}$.

Remark 2.26. *Theorem 2.24 offers us a way to directly see the validity of Lemma 2.14 with compactness added to the requirements. If all λ are nonnegative in* (2.4)*, then we can construct a square root of Q by replacing all λ with $\sqrt{\lambda}$ in the equation.*

Two more useful facts about spectral theory for which the proofs can be found in [12, Theorem 8.71] and [12, Theorem 8.52] respectively are the following.

Lemma 2.27. *Let X be a Banach space and $Q\colon X \to X$ a bounded linear operator. Then $\sigma(Q_\lambda) \neq \emptyset$*

Lemma 2.28. *Let X be a complex Hilbert space and $Q\colon X \to X$ a self-adjoint linear operator. Then $\sigma(Q_\lambda) \subseteq \mathbb{R}$.*

Another important result for us is the following slightly weakened version of [8, V. Theorem 4.10], which states that a bounded and self-adjoint operator is stable in the sense that a small (with respect to the operator norm) self-adjoint perturbation has only little effect on the spectrum. We will use this theorem to prove a continuity property between the spectra of a whole family of operators, though.

Theorem 2.29. *Let H be a complex Hilbert space and T, A be bounded self-adjoint linear operators from H onto itself.*
Then $S = T + A$ is self-adjoint and $dist(\sigma(S), \sigma(T)) \leq \|A\|$, that is,

$$\sup_{\lambda\in\sigma(S)} dist(\lambda, \sigma(T)) \leq \|A\|, \qquad \sup_{\lambda\in\sigma(T)} dist(\lambda, \sigma(S)) \leq \|A\|$$

A proof can be found in [8].

2.2.2 Complexification

Imagine the situation in linear algebra, where we have the real matrix

$$A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

It's characteristical polynomial is $\lambda^2 + 1$ which has no real root. One might say, A has i and $-i$ as eigenvalues because

$$A\begin{pmatrix} i \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} = -i\begin{pmatrix} i \\ -1 \end{pmatrix} \qquad A\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix} = i\begin{pmatrix} i \\ 1 \end{pmatrix}$$

However if we regard A as an endomorphism on $\mathbb{R}^2$, then i and $-i$ are clearly no eigenvalues of A because not only are they not in $\mathbb{R}$, but the eigenvectors corresponding to them are not even in $\mathbb{R}^2$. By saying i and $-i$ are eigenvalues of A, we implicitly expand the space from $\mathbb{R}^2$ to $\mathbb{C}^2$ and a similar procedure can be done in a setting with vector spaces of infinite dimension.

I wrote this section based on [1, Section 9.14] for two reasons. Firstly we need Theorem 2.29 in a real setting but [8] only develops the complex case. Secondly this topic was indeed a hurdle for me when reading the research papers that this thesis is based on. The authors often take for example real Banach spaces, but then regard the spectrum of operators between these spaces as being in the complex plane. Hopefully the following pages will clear up how we can apply aspects of spectral theory developed for complex function spaces to real function spaces even if the proofs are mostly skipped. We limit ourselves to the case of Hilbert spaces because that is what we will work in later.

Definition 2.30. *Let H be a real Hilbert space, we define it's complexification $\tilde{H} := H \times H$ with the usual componentwise addition on $H \times H$.*

Remark 2.31. *Always keep in mind that elements $(h_1, h_2) \in \tilde{H}$ could be denoted by $h_1 + ih_2$ if we would strive for less formal correctness. By visualizing them like that one can easily see how the definitions of this section are motivated. After defining scalar multiplication we will see that this unformal notation would be not so bad after all due to $(h_1, h_2) = (h_1, 0) + i(0, h_2)$.*

Definition 2.32. *Let $(h_1, h_2), (h_3, h_4) \in \tilde{H}$. We equip $\tilde{H}$ with the following scalar multiplication and complex conjugate map. For $\alpha \in \mathbb{C}$ we can uniquely write $\alpha = a + ib$ with $a, b \in \mathbb{R}$.*

$$\begin{aligned} \alpha(h_1, h_2) &:= (ah_1 - bh_2, ah_2 + bh_1) \\ \overline{(h_1, h_2)} &:= (h_1, -h_2) \end{aligned}$$

Furthermore we define an inner product on $\tilde{H}$ through

$$\langle (h_1, h_2), (h_3, h_4) \rangle_{\tilde{H}} := \langle h_1, h_3 \rangle_H + \langle h_2, h_4 \rangle_H + i\langle h_2, h_3 \rangle_H - i\langle h_1, h_4 \rangle_H$$

We skip the easy calculations that what we defined is indeed a scalar multiplication and an inner product respectively and observe

$$\begin{aligned} \|(h_1, h_2)\|_{\tilde{H}}^2 &= \langle (h_1, h_2), (h_1, h_2) \rangle_{\tilde{H}} \\ &= \langle h_1, h_1 \rangle_H + \langle h_2, h_2 \rangle_H + i\langle h_2, h_1 \rangle_H - i\langle h_1, h_2 \rangle_H \\ &= \|h_1\|_H^2 + \|h_2\|_H^2 \end{aligned} \tag{2.5}$$

by using the symmetry of the inner product in H. If we have a Cauchy sequence in $\tilde{H}$ we see by (2.5) that both the first and the second component define Cauchy sequences in H and as such converge. Arranging their two limits as an element of $\tilde{H}$, one sees by (2.5) that the Cauchy sequence in $\tilde{H}$ converges. Hence $\tilde{H}$ is complete and in fact a proper complex Hilbert space.

Definition 2.33. *Let $Q\colon H \to H$ be a linear operator. We define $\tilde{Q}\colon \tilde{H} \to \tilde{H}$ through*

$$\tilde{Q}(h_1, h_2) := (Qh_1, Qh_2)$$

for $(h_1, h_2) \in \tilde{H}$.

Lemma 2.34. *$\tilde{Q}\colon \tilde{H} \to \tilde{H}$ is a linear operator.*
We have the following implications:

- *If Q is bounded, then $\tilde{Q}$ is bounded and $\|Q\| = \|\tilde{Q}\|$.*
- *If Q is compact, then $\tilde{Q}$ is compact.*
- *If Q is self-adjoint, then $\tilde{Q}$ is self-adjoint.*

Proof. The linearity and the part about self-adjointness are left to the reader as an exercise.

If Q is bounded, then by (2.5) we have for any $(h_1, h_2) \in \tilde{H}$

$$\begin{aligned}\|\tilde{Q}(h_1, h_2)\|_{\tilde{H}}^2 &= \|(Qh_1, Qh_2)\|_{\tilde{H}}^2 = \|Qh_1\|_H^2 + \|Qh_2\|_H^2 \\ &\leq \|Q\|\|h_1\|_H^2 + \|Q\|\|h_2\|_H^2 = \|Q\|\|(h_1, h_2)\|_{\tilde{H}}^2\end{aligned}$$

which means that $\tilde{Q}$ is bounded and $\|\tilde{Q}\| \leq \|Q\|$.
Now $\|\tilde{Q}\| \geq \|Q\|$ is because for all $x \in H$ we have

$$\begin{aligned}\sqrt{2}\|Qx\|_H &= \|(Qx, Qx)\|_{\tilde{H}} = \|\tilde{Q}(x, x)\|_{\tilde{H}} \\ &\leq \|\tilde{Q}\|\|(x, x)\|_{\tilde{H}} = \|\tilde{Q}\|\sqrt{2}\|x\|_H\end{aligned}$$

Let Q be compact and $((x_n, y_n))_{n\in\mathbb{N}} \subset \tilde{H}$ be a bounded sequence. $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are bounded sequences in H through (2.5). Due to the compactness of Q, $(Qx_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(Qx_{n_k})_{k\in\mathbb{N}}$ and $(Qy_{n_k})_{k\in\mathbb{N}}$ also has a convergent subsequence. If we denote their respective limits by x_∞ and y_∞, then by (2.5) we see that there exists a subsequence of $(\tilde{Q}(x_n, y_n))_{n\in\mathbb{N}} = ((Qx_n, Qy_n))_{n\in\mathbb{N}}$ that converges to (x_∞, y_∞) with respect to $\|.\|_{\tilde{H}}$ and hence $\tilde{Q}$ is compact. □

Lemma 2.35. *The real eigenvalues of $\tilde{Q}$ and the eigenvalues of Q are the same.*

Proof. Let $\lambda \in \mathbb{R}$ be an eigenvalue of Q. Then there exists a $h \in H$, $h \neq 0$ that is an eigenvector of Q to λ. Through

$$\tilde{Q}(h, 0) = (Qh, Q0) = (\lambda h, 0) = \lambda(h, 0)$$

and $(h, 0) \neq 0$, λ is an eigenvalue of $\tilde{Q}$.

Conversely let $\tau \in \mathbb{R}$ be a real eigenvalue of $\tilde{Q}$. There is an eigenvector $(h_1, h_2) \in \tilde{H}$, $(h_1, h_2) \neq 0$ to τ. Because τ is real we have

$$(Qh_1, Qh_2) = \tilde{Q}(h_1, h_2) = \tau(h_1, h_2) = (\tau h_1, \tau h_2)$$
$$\implies Qh_1 = \tau h_1,\ Qh_2 = \tau h_2$$

and due to $(h_1, h_2) \neq 0$ either $h_1 \neq 0$ or $h_2 \neq 0$. Hence τ is an eigenvalue of Q. □

This means that if we can gain information on the real eigenvalues of $\tilde{Q}$ through any means, then we also have the same information for the eigenvalues of Q.

2.3 Riesz' representation theorem

In this section we will prove a modification of Riesz' representation theorem for bilinear forms that we will later in this thesis need. We start with Riesz' well-known representation theorem in it's usual form, which is in this case taken from [15, Section III.6] where one can also find it's proof. Note that unlike [15] we write $\langle .,. \rangle$ for inner products.

Lemma 2.36. *Let H be a Hilbert space and b a bounded linear functional on H. Then there exists a uniquely determined vector y_b of H such that*

$$b(v) = \langle v, y_b \rangle \text{ for all } v \in H, \text{ and } \|b\| = \|y_b\|$$

Conversely, any vector $\Psi \in H$ defines a bounded linear functional a_Ψ on H by

$$a_\Psi(v) = \langle v, \Psi \rangle \text{ for all } v \in H, \text{ and } \|a_\Psi\| = \|\Psi\|$$

What [15] calls a 'bounded linear functional' is simply an element of H^*, the dual space of H. Note that Lemma 2.36 is valid for both real and complex Hilbert spaces, hence our now following modified version for bilinear forms is also valid for sesquilinear forms in complex Hilbert spaces. But because later we will only need it for real Hilbert spaces, we drop the complex case to slightly ease up notation.

Theorem 2.37. *Let H be a real Hilbert space and $b\colon H \times H \to \mathbb{R}$ a bounded bilinear functional, i.e. b needs to satisfy the following conditions:*

$$\begin{aligned} b(\alpha v_1 + v_2, \Psi) &= \alpha b(v_1, \Psi) + b(v_2, \Psi) & \\ b(v, \alpha\Psi_1 + \Psi_2) &= \alpha b(v, \Psi_1) + b(v, \Psi_2) & \textit{Bilinearity} \\ |b(v, \Psi)| &\leq \gamma \|v\| \|\Psi\| & \textit{Boundedness} \end{aligned}$$

for all $v, v_1, v_2, \Psi, \Psi_1, \Psi_2 \in H$, all $\alpha \in \mathbb{R}$ and some $\gamma > 0$.
Then there exists a bounded linear operator $B\colon H \to H$ such that:

$$b(v, \Psi) = \langle Bv, \Psi \rangle \quad \forall\ v, \Psi \in X$$

Proof. If we look at the situation for a fixed $v \in H$, we observe that $b_v := b(v, .)$ satisfies the conditions of Lemma 2.36. Consequentially for every $v \in H$ we get a $y_{b_v} \in H$ such that

$$b(v, \Psi) = b_v(\Psi) = \langle y_{b_v}, \Psi \rangle \quad \forall\ \Psi \in H$$

Now we define $B\colon H \to H$ to map v onto y_{b_v} and easily see from the above equation that

$$b(v, \Psi) = \langle Bv, \Psi \rangle \quad \forall\ v, \Psi \in H \tag{2.6}$$

All we have to do now is to verify that B is linear and bounded. Let $v_1, v_2, \Psi \in H$ and observe that due to the bilinearity of b and (2.6) we can calculate

$$\begin{aligned} \langle Bv_1 + Bv_2, \Psi \rangle &= \langle Bv_1, \Psi \rangle + \langle Bv_2, \Psi \rangle = b_{v_1}(\Psi) + b_{v_2}(\Psi) \\ &= b(v_1, \Psi) + b(v_2, \Psi) = b(v_1 + v_2, \Psi) = b_{v_1+v_2}(\Psi) \\ &= \langle B(v_1 + v_2), \Psi \rangle \end{aligned} \tag{2.7}$$

Since (2.7) is valid for all $\Psi \in H$, we can conclude $Bv_1 + Bv_2 = B(v_1 + v_2)$ by shifting everything to the left hand side of the equation and then setting $\Psi := Bv_1 + Bv_2 - B(v_1 + v_2)$. Adding to that the similarly calculated multiplicativity

$$\begin{aligned} \forall\ \Psi \in H : \quad \langle B(\alpha v), \Psi \rangle &= b_{\alpha v}(\Psi) = b(\alpha v, \Psi) \\ &= \alpha b(v, \Psi) = \alpha b_v(\Psi) = \alpha \langle Bv, \Psi \rangle \\ &= \langle \alpha Bv, \Psi \rangle \\ \implies B(\alpha v) &= \alpha Bv \end{aligned}$$

we see that B is indeed linear. For the remaining boundedness of B we observe that for all $v \in H$

$$\begin{aligned} \|Bv\|^2 &= \langle Bv, Bv \rangle = b_v(Bv) = b(v, Bv) \\ &\leq \gamma \|v\| \|Bv\| \\ \implies \|Bv\| &\leq \gamma \|v\| \end{aligned}$$

due to the boundedness of b, which concludes the proof. □

2.4 On function spaces

Having these tools established it is now time to take a closer look at the most important function space for this thesis, the Sobolev space $H_0^2(D)$. We already mentioned it in Definition 1.2 without further explaining, what kind of functions it encompasses. In order to do so we need a little bit of background which will be given in this section. The general approach to Sobolev spaces presented here is a very direct one adopted from [2], netting us the advantage to be easily able to imagine what a function in the occurring Sobolev spaces looks like. A downside to this approach is that within our context we are not able to treat distributions which are not 'nice' in the sense that they can be identified with a L^2 function. However we are not interested in any of these so this will not pose a problem. Should the reader be interested in a more general approach on distributions and Sobolev spaces I recommend [12]. As mentioned before, we only cover real valued function spaces.

2.4.1 $L^2(D)$ and $C_0^\infty(D)$

First we want to set the starting point on our quest to understand what the $H_0^2(D)$ looks like.

Definition 2.38. *$L^2(D)$ is a set of equivalency classes of functions. Given a function $f\colon D \to \mathbb{R}$ satisfying*

$$\|f\|_{L^2(D)} := \sqrt{\iint_D |f(x)|^2 \, dx} < \infty \tag{2.8}$$

we define it's equivalency class to be

$$[f] := \left\{ g\colon D \to \mathbb{R} \mid \|f - g\|_{L^2(D)} = 0 \right\}$$

$L^2(D)$ is the set of all equivalency classes of functions satisfying (2.8)*.*

$$\begin{aligned} L^2(D) &:= \left\{ [f] \mid f\colon D \to \mathbb{R}, \|f\|_{L^2(D)} < \infty \right\} \\ \|[f]\|_{L^2(D)} &:= \|f\|_{L^2(D)} \\ [f] + [g] &:= [f + g] \\ \alpha \cdot [f] &:= [\alpha \cdot f] \end{aligned}$$

This is standard analysis. The reason why we have to take equivalency classes is that the norm as defined above would not be positive definite otherwise. Following convention we will drop the distinction between functions and equivalency classes and restrict ourselves to equalities 'almost everywhere' (in D) or equivalently 'for almost all $x \in D$' as we have already done in Definition 1.1 for example. This means that the equalities are allowed to be untrue on any set of Lebesgue measure zero for fixed representative functions of the equivalency classes. Such statements make sense because for $f = g$ almost everywhere we have $\|f - g\|_{L^2(D)} = 0$ and hence $f = g$ in the L^2-sense.

We recall the following fact from [12, Example 6.23].

Lemma 2.39. *$L^2(D)$ equipped with*

$$\langle x, y\rangle_{L^2(D)} := \iint\limits_D x(t)\cdot y(t)\ dt$$

is a Hilbert space, in particular $\langle .,.\rangle_{L^2(D)}$ is an inner product.

Another important function space besides $L^2(D)$ is $C_0^\infty(D)$. We assume that the concepts of continuity and (classical) differentiation are known and the corresponding notation becomes clear from the context. The overline in the following definition denotes the closure with respect to the usual metric on $\mathbb{R}^2$ and for our purposes the 0 is always included in $\mathbb{N}$.

Definition 2.40. *$\overline{\{x \in D \mid f(x) \neq 0\}} \cap D$ is called the support of f or supp(f).*

$$\begin{aligned} C(D) &:= \{f\colon D \to \mathbb{R} \mid f \text{ is continuous}\} \\ C^\infty(D) &:= \left\{ f \in C(D) \mid \frac{\partial^{n+m} f}{\partial x_1^n \partial x_2^m}\ exists\ for\ all\ n, m \in \mathbb{N} \text{ and is continuous} \right\} \end{aligned}$$

Definition 2.41. *Functions in the following set are called test functions.*

$$C_0^\infty(D) := \{f \in C^\infty(D) \mid supp(f) \text{ is compact}\}$$

Since D is bounded, the condition that f has compact support is equivalent to $\overline{\{x \in D \mid f(x) \neq 0\}} \subseteq D$. Furthermore any $f \in C_0^\infty(D)$ is clearly in $L^2(D)$ because as a continuous function f is bounded on the compact set *supp*(f) and

$$\|f\|^2_{L^2(D)} = \iint\limits_D |f(x)|^2\ dx \leq \iint\limits_D \|f\|^2_{L^\infty(D)}\ dx$$

where the right hand side is less or equal than $\|f\|^2_{L^\infty(D)}$ times the finite Lebesgue measure of D. And as we just implicitly used it yet again, here is the formally correct definition for another function space from the introduction.

Definition 2.42.

$$\begin{aligned} \|f\|_{L^\infty(D)} &:= \inf\{C > 0 \mid f \leq C \text{ almost everywhere in } D\} \\ L^\infty(D) &:= \left\{ f \in L^2(D) \mid \|f\|_{L^\infty(D)}\ is\ finite \right\} \end{aligned}$$

2.4.2 $H^1(D)$ and $H^2(D)$

Usually the desire to differentiate functions that are not differentiable in the classic sense leads to the notion of distributions, elements of the dual space of the space of test functions. This rather abstract concept can then be extended to a theory where one can differentiate without restrictions and identify the distributions with classical functions in some sense. The idea behind this revolves to a huge deal around the integration by parts. Our approach to 'weak' derivatives also uses integration by parts, but is less ambitious in the sense that we will not be able to differentiate any L^2 function at the end. This section is based on [2].

Definition 2.43. *Let $u, v \in L^2(D)$ and $i, j \in \mathbb{N}$. If*

$$\iint_D u \frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} \Psi \, dx = (-1)^{i+j} \iint_D v\Psi \, dx \tag{2.9}$$

is true for all $\Psi \in C_0^\infty(D)$, then we call v the $(i,j)^{th}$ weak derivative of u and write

$$v = \frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} u = \frac{\partial^{i+j} u}{\partial x_1^i \partial x_2^j}$$

One observes that we used the same notation for weak derivatives as we did for classical derivatives. To understand why this makes sense we need to know two things.

Lemma 2.44. *Weak derivatives in the sense of Definition 2.43 are unique in $L^2(D)$ if they exist.*

For a proof of this see [2, Lemma 6.1.4]. The second thing we need to know is [2, Lemma 6.1.5].

Lemma 2.45. *If $v \in L^2(D)$ is the $(i,j)^{th}$ derivative of $u \in L^2(D)$ in the classical sense, then (2.9) holds true for all $\Psi \in C_0^\infty(D)$ and hence v is also the $(i,j)^{th}$ weak derivative of u.*

This can be seen from integration by parts because every boundary term involving a test function vanishes due to the compact support of the test functions. Due to the two previous lemmata we will not make a distinction between classical and weak derivatives anymore in the future. Also we are in a position where we can define our first Sobolev spaces.

Definition 2.46. *$H^2(D)$ is the set of all functions $u \in L^2(D)$ such that*

$$\frac{\partial u}{\partial x_1}, \quad \frac{\partial u}{\partial x_2}, \quad \frac{\partial^2 u}{\partial x_1^2}, \quad \frac{\partial^2 u}{\partial x_2^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial x_1 \partial x_2}$$

all exist in $L^2(D)$.

$$\|u\|_2 := \sqrt{\iint_D (u)^2 + (\frac{\partial u}{\partial x_1})^2 + (\frac{\partial u}{\partial x_2})^2 + (\frac{\partial^2 u}{\partial x_1^2})^2 + (\frac{\partial^2 u}{\partial x_2^2})^2 + (\frac{\partial^2 u}{\partial x_1 \partial x_2})^2 \, dx}$$

Definition 2.47. *$H^1(D)$ is the set of all functions $u \in L^2(D)$ such that*

$$\frac{\partial}{\partial x_1}u \quad \textit{and} \quad \frac{\partial}{\partial x_2}u$$

both exist in $L^2(D)$.

$$\|u\|_1 := \sqrt{\iint\limits_D (u)^2 + (\frac{\partial u}{\partial x_1})^2 + (\frac{\partial u}{\partial x_2})^2}$$

Observe that $H^1(D)$ and $H^2(D)$ are vector spaces and that $\|.\|_1$ and $\|.\|_2$ indeed describe a norm on 'their' space. Everything except the triangle inequality can be seen without much hassle, and the triangle inequality itself follows from the triangle inequality in $L^2(D)$ with an easy calculation. We do not prove these things here in detail, but instead cite the following lemma.

Lemma 2.48. *$H^2(D)$ equipped with the norm $\|.\|_2$ is a Banach space. The same holds true for $H^1(D)$ equipped with $\|.\|_1$.*

This is well-known, for a proof see e.g. [2, Theorem 6.2.3]. Continuing from the definitions one can already suspect that we could also denote $L^2(D)$ by $H^0(D)$. We will not do this, but it will be of some importance for an imbedding theorem that we will cite later. Another thing one can easily see from the definitions is that any function in $H^2(D)$ is also in $H^1(D)$. With this in mind we cite a special case of [12, Theorem 7.32] and refer the reader to [12] for a proof.

Lemma 2.49. *There exists $c > 0$ such that for all $u \in H^1(D)$ we have*

$$\|u\|_1^2 \leq c \iint\limits_D (\frac{\partial u}{\partial x_1})^2 + (\frac{\partial u}{\partial x_2})^2 \, dx = c \iint\limits_D \nabla u \cdot \nabla u \, dx$$

Definition 2.50. *Let $u \in H^2(D)$. Then*

$$\Delta u := \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \qquad \textit{and} \qquad \nabla u := \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \end{pmatrix}$$

2.4.3 $H_0^2(D)$

Because we can easily see that any test function is in $H^2(D)$ from the definitions, the following makes sense.

Definition 2.51. *$H_0^2(D)$ is the closure of $C_0^\infty(D)$ in $H^2(D)$ with respect to $\|.\|_2$.*

So we have to imagine $H_0^2(D)$ as those functions in $L^2(D)$ whose weak derivatives up to order two exist in $L^2(D)$ and who are the $H^2(D)$-limit of a sequence of test functions.

Remark 2.52. *$H_0^2(D)$ equipped with $\|.\|_2$ is a Banach space [2, Definition 6.2.11].*

Lemma 2.53. *$H_0^2(D)$ is dense in $L^2(D)$. Hence if $v \in L^2(D)$ and*

$$0 = \iint\limits_D v\Psi \; dx \quad \forall \; \Psi \in H_0^2(D)$$

then $v = 0$.

Proof. By construction we have $C_0^\infty(D) \subseteq H_0^2(D)$. Furthermore $C_0^\infty(D)$ is dense in $L^2(D)$, see e.g. [14, Lemma VI.1.10] for a proof of this. For the second part observe that for any $\Psi \in H_0^2(D)$ with the help of the Cauchy-Schwarz inequality (Lemma 2.4) for $L^2(D)$

$$\begin{aligned} 0 &= \iint\limits_D v\Psi \; dx = \iint\limits_D v(\Psi - v + v) \; dx = \|v\|_{L^2(D)}^2 + \iint\limits_D v(\Psi - v) \; dx \\ &\geq \|v\|_{L^2(D)}^2 - \|v\|_{L^2(D)} \|\Psi - v\|_{L^2(D)} \end{aligned}$$

If $\|v\|_{L^2(D)}$ were not equal to zero, then above calculation would be a contradiction because we can make $\|\Psi - v\|_{L^2(D)}$ arbitrarily small by virtue of the density part of the lemma. ☐

A question that remains open is how the boundary conditions

$$u = w, \; \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} \; on \; \partial D \iff v = 0, \; \frac{\partial v}{\partial \nu} = 0 \; on \; \partial D \tag{2.10}$$

for $v := u - w$ in Definition 1.2 have to be understood. While we know that $v \in H_0^2(D)$ is indeed continuous (see e.g. [12, Theorem 7.27]) and thus could be evaluated on the boundary by evaluating it's continuous extension to it, the case is not as easy for it's normal derivative.

In general the approach to this issue to find a continuous trace operator from a 'non-smooth' function space to a suitable space of functions on the boundary that behaves just like restriction to the boundary for any functions where that makes sense. Similarly we can search for a continuous operator that gives a 'non-smooth' function a generalized normal derivative on the boundary that is the same as the usual normal derivative whenever the latter exists. [2, Theorem 6.3.11] states that such an operator exists for the space $H_0^2(D)$.

However we do not need to go into details because of [12, Theorem 7.41] which we slim down and cite in the form of the following lemma.

Lemma 2.54. *$H_0^2(D)$ is the set of all functions $v \in H^2(D)$ such that*

$$v = 0,\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial D$$

in the sense of trace.

Or in other words for any $v \in H_0^2(D)$ both v and $\frac{\partial v}{\partial \nu}$ are zero on the boundary ∂D in the sense of trace under our smoothness assumptions for ∂D. Hence the demand for (2.10) in Definition 1.2 is in fact redundant if we already demand $u-w \in H_0^2(D)$. In the central chapters 3 and 4 we will always work with the latter condition and Definition 2.51 while keeping (2.10) only around as a visual aid to imagine how the elements of $H_0^2(D)$ look like.

Green's second formula is not only valid in $H_0^2(D)$, but takes an especially manageable form due to the boundary conditions from Lemma 2.54 which are included in $H_0^2(D)$.

Theorem 2.55. *Let $v, \Psi \in H_0^2(D)$. Then*

$$\iint\limits_D v\Delta\Psi\ dx = \iint\limits_D \Psi\Delta v\ dx$$

As one can easily see, the theorem above is a corollary of the following lemma.

Lemma 2.56. *Let $v, \Psi \in H_0^2(D)$. Then*

$$\iint\limits_D v\Delta\Psi\ dx = \iint\limits_D \nabla v \cdot \nabla\Psi\ dx$$

This is [2, equation 6.6.4] where we plugged in our $H_0^2(D)$ functions and kept in mind Lemma 2.54. For a proof of this equation the reader is referred to [2].

Now we equip the $H_0^2(D)$ with a custom inner product suited for our calculations, this will then be the function space that we will mostly work with. The following theorem is strongly inspired by [10, Section 4.5] where something similar is proven in a slightly different setting.

Theorem 2.57. *$H_0^2(D)$ equipped with the inner product*

$$\langle x, y\rangle_{H_0^2(D)} := \iint\limits_D \Delta x\Delta y\ \frac{dx}{q-p} \tag{2.11}$$

with p, q as in Definition 1.1 is a Hilbert space. The norm

$$\|u\|_{H_0^2(D)} := \sqrt{\langle u, u\rangle_{H_0^2(D)}}$$

is equivalent to the usual norm on $H_0^2(D)$ from Definition 2.46, $\|.\|_2$.
i.e. there exist constants $c > 0$ and $C > 0$ such that

$$c\|u\|_2 \leq \|u\|_{H_0^2(D)} \leq C\|u\|_2 \quad \forall\, u \in H_0^2(D) \tag{2.12}$$

Proof. Recall Definition 2.2 and observe that linearity, symmetry and positive semi-definiteness of $\langle .,. \rangle_{H_0^2(D)}$ can easily be seen. Assume for the moment that (2.12) is valid. Since $H_0^2(D)$ equipped with $\|.\|_2$ is a Banach space (Remark 2.52), the rest of the theorem is an easy corollary of (2.12) as follows:

Because $\|.\|_2$ is positive definite, so is $\langle .,. \rangle_{H_0^2(D)}$ and hence the latter is indeed an inner product. Furthermore a Cauchy sequence (recall Definition 2.6) with respect to $\|.\|_{H_0^2(D)}$ is also a Cauchy sequence with respect to $\|.\|_2$ by (2.12). But then a limit exists in $H_0^2(D)$ with respect to the norm $\|.\|_2$ and, again applying (2.12), we see that the sequence also converges to that limit with respect to $\|.\|_{H_0^2(D)}$.

So all we have to do is to check norm equivalency (2.12). Let $u \in H_0^2(D)$. One direction can be seen from the Cauchy-Schwarz inequality (Lemma 2.4) for $L^2(D)$ in which derivatives of u up to second order are by Definition 2.46. We recall $q - p \geq q_0$ from Definition 1.1 and calculate

$$\begin{aligned}
\|u\|^2_{H_0^2(D)} &= \iint_D (\Delta u)^2 \,\frac{dx}{q-p} \leq \iint_D (\Delta u)^2 \,\frac{dx}{q_0} \\
&= \frac{1}{q_0} \iint_D \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right)^2 dx \\
&= \frac{1}{q_0} \iint_D \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 dx + \frac{2}{q_0} \iint_D \left(\frac{\partial^2 u}{\partial x_1^2}\right)\left(\frac{\partial^2 u}{\partial x_2^2}\right) dx \\
&\leq \frac{1}{q_0} \iint_D \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 dx + \frac{2}{q_0} \sqrt{\iint_D \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 dx} \sqrt{\iint_D \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 dx} \\
&\leq \frac{1}{q_0}\|u\|_2^2 + \frac{2}{q_0}\|u\|_2\|u\|_2 = \frac{3}{q_0}\|u\|_2^2
\end{aligned}$$

Taking the square root on both sides of the equation gives us one half of (2.12).

For the other half we have to dig a little deeper and cite a regularity result for elliptic partial differential equations. In particular we can use [12, Theorem 9.53] to get

$$\|u\|_2 \leq c_1(\|u\|_{L^2(D)} + \|\Delta u\|_{L^2(D)}) \tag{2.13}$$

for all $u \in H_0^2(D)$ and some $c_1 > 0$.
Although it is a bit awkward here because we lack context, let's quickly go over

the details on how we arrive at (2.13). Note that $u \in H_0^2(D)$ is a solution $\tilde{u}$ to the Dirichlet problem

$$-\Delta\tilde{u} = -\Delta u \quad in\ D \qquad\qquad \tilde{u} = 0 \quad on\ \partial D$$

The boundary condition is fulfilled due to Lemma 2.54 and the interior condition is obvious. Furthermore $-\Delta$ is uniformly elliptic [12, Example 9.3]. Now [12, Theorem 9.53] is a general result for solutions to this kind of problems and gives us (2.13).

From Lemma 2.49 we have

$$\|u\|_1^2 \leq c_2 \iint_D \nabla u \cdot \nabla u \; dx \tag{2.14}$$

for all $u \in H_0^2(D)$ and some $c_2 > 0$.

Now we can conclude the following from the definition of $\|.\|_1$, (2.14), Lemma 2.56 and yet again the Cauchy-Schwarz inequality for $L^2(D)$.

$$\|u\|_{L^2(D)}^2 \leq \|u\|_1^2 \leq c_2 \iint_D \nabla u \cdot \nabla u \; dx = -c_2 \iint_D u\Delta u \; dx \leq c_2\|u\|_{L^2(D)}\|\Delta u\|_{L^2(D)}$$

Dividing this by $\|u\|_{L^2(D)}$ and plugging it into (2.13) we get

$$\|u\|_2 \leq c_1c_2\|\Delta u\|_{L^2(D)} + c_1\|\Delta u\|_{L^2(D)} = c_1(1+c_2)\|u\|_{H_0^2(D)}$$

which concludes the proof. □

A corollary of Theorem 2.57 is the following lemma.

Lemma 2.58. *There exists some $c > 0$ with*

$$\iint_D (\Delta x)^2 \, \frac{dx}{q-p} \geq c \iint_D x^2 \, \frac{dx}{q-p} \quad \textit{for all } x \in H_0^2(D)$$

Proof. Let $x \in H_0^2(D)$. Then

$$\begin{aligned}\iint_D x^2 \, \frac{dx}{q-p} \leq \iint_D x^2 \, \frac{dx}{q_0} = \frac{1}{q_0}\|x\|_{L^2(D)} \leq \frac{1}{q_0}\|x\|_2 \leq \frac{1}{q_0}C\|x\|_{H_0^2(D)} \\ = \frac{1}{q_0}C \iint_D (\Delta x)^2 \, \frac{dx}{q-p}\end{aligned}$$

where $C > 0$ is the constant from Theorem 2.57. □

2.5 Compact imbeddings

The first step in this section is to give a definition of what a compact imbedding is and the second is to cite a couple of particular compact imbedding properties for the Sobolev spaces we defined earlier. Because we defined these spaces to be real, we restrict ourselves to the real case. In a third step we will then discuss how these compact imbeddings will help us later.

We start by giving two basic definitions which are slightly modified versions of their counterparts [12, Definition 7.15] and [12, Definition 7.25] respectively.

Definition 2.59. *Let X and Y be real Banach spaces. X is said to be continuously imbedded in Y, if $X \subseteq Y$ and there exists a constant $c > 0$ such that*

$$\|u\|_Y \leq c\|u\|_X$$

for all $u \in X$. In this case we write $X \hookrightarrow Y$.

Definition 2.60. *Let X and Y be Banach spaces such that $X \hookrightarrow Y$. X is said to be compactly imbedded in Y, if every bounded sequence in X has a convergent subsequence in Y. In this case we write $X \overset{c}{\hookrightarrow} Y$.*

Equivalently we could say that the naturally defined identity $I\colon X \to Y$ mapping every $x \in X$ onto itself is a compact operator in the sense of Definition 2.15.

Lemma 2.61. *Let X, Y, Z be Banach spaces such that $X \hookrightarrow Y \hookrightarrow Z$.*
Then $X \hookrightarrow Z$. If additionally $X \hookrightarrow Y \overset{c}{\hookrightarrow} Z$, then $X \overset{c}{\hookrightarrow} Z$.

Proof. $X \subseteq Y \subseteq Z$ implies $X \subseteq Z$. The continuity of the imbedding of X into Z is proven quickly as well. Due to $X \hookrightarrow Y$ there exists $c_1 > 0$ such that $\|u\|_Y \leq c_1\|u\|_X$ for all $u \in X$, and because $Y \hookrightarrow Z$, there also exists $c_2 > 0$ such that $\|u\|_Z \leq c_2\|u\|_Y$ for all $u \in Y$. Then

$$\|u\|_Z \leq c_2\|u\|_Y \leq c_2 c_1\|u\|_X$$

is true for all $u \in X$, or in other words $X \hookrightarrow Z$.

Now let $Y \overset{c}{\hookrightarrow} Z$. Observe that any sequence that is bounded in X is also bounded in Y because of $X \hookrightarrow Y$. Hence it has a convergent subsequence in Z due to $Y \overset{c}{\hookrightarrow} Z$. This means $X \overset{c}{\hookrightarrow} Z$ and concludes the proof. □

Since Theorem 2.57 states that $\|.\|_{H_0^2(D)}$ and $\|.\|_2$ are equivalent norms on $H_0^2(D)$, we have

$$(H_0^2(D), \|.\|_{H_0^2(D)}) \hookrightarrow (H_0^2(D), \|.\|_2)$$
$$(H_0^2(D), \|.\|_2) \hookrightarrow (H_0^2(D), \|.\|_{H_0^2(D)})$$

Hence through Lemma 2.61

$$(H_0^2(D), \|.\|_{H_0^2(D)}) \hookrightarrow X \iff (H_0^2(D), \|.\|_2) \hookrightarrow X$$
$$(H_0^2(D), \|.\|_{H_0^2(D)}) \overset{c}{\hookrightarrow} X \iff (H_0^2(D), \|.\|_2) \overset{c}{\hookrightarrow} X$$

where X is any Banach space. Because of that we can drop the cumbersome notation including the norm and write simply $H_0^2(D)$ within our imbedding properties.

Lemma 2.62. $H_0^2(D) \hookrightarrow H^2(D)$

Proof. $H_0^2(D)$ is a subset of H^2 and both use the same norm $\|.\|_2$. Hence the lemma is clear from Definition 2.59. □

The following lemma is also known as the Rellich imbedding theorem. We cite it from [12, Theorem 7.29], where one can also find it's proof.

Lemma 2.63. *Let $k \in \mathbb{N}$. Then*

$$H^{k+1}(D) \overset{c}{\hookrightarrow} H^k(D)$$

And with that we can give the following lemma which is the reason why we introduced compact imbeddings.

Lemma 2.64. $H_0^2(D) \overset{c}{\hookrightarrow} L^2(D)$

Proof. Because $H^0(D)$ is the same as $L^2(D)$, we have from our lemmata 2.62 and 2.63

$$H_0^2(D) \hookrightarrow H^2(D) \overset{c}{\hookrightarrow} H^1(D) \overset{c}{\hookrightarrow} L^2(D)$$

Applying Lemma 2.61 twice concludes the proof. □

Note that the assumptions on D given in Definition 1.1 are valid throughout this entire thesis and that in a more general setting not all of the aforementioned lemmata hold true.

There is still one question open for this section, namely how Lemma 2.64 will be of use later. The answer comes in the form of yet another lemma.

Lemma 2.65. *Let X and Y be Hilbert spaces such that $X \overset{c}{\hookrightarrow} Y$.*
Then $\|x_n\|_Y \to 0$ for any $(x_n)_{n\in\mathbb{N}} \subset X$ that converges weakly to zero in X.

Proof. This is a direct corollary of Lemma 2.19. $X \overset{c}{\hookrightarrow} Y$ implies that the identity operator $I\colon X \to Y, x \mapsto x$ is compact (Compare Definition 2.60 and Definition 2.15). Applying Lemma 2.19 to $I\colon X \to Y$ concludes the proof. □

Chapter 3

Countability

In the following chapter we want to show that the real positive interior transmission eigenvalues in the sense of Definition 1.2 are countable and have no accumulation point other than possibly infinity. This means that they are isolated points and there can not be for example a whole interval of them. A first section will be devoted to the construction of a linear operator between Hilbert spaces such that any interior transmission eigenvalues are also eigenvalues of this operator. After that is accomplished, we will in a second section narrow down the set of possible eigenvalues for that operator with the tools established earlier. Throughout this chapter we closely orientate ourselves along [9], where the case $p = 0$ is covered and mimic most of the proofs from there while adding detail.

3.1 Characterisation

We need the following characterisation of interior transmission eigenvalues which is an extended version of a result in [9].

Theorem 3.1. *$\lambda > 0$ is an interior transmission eigenvalue in the sense of Definition 1.2 if and only if the following statement is true:*

$$\exists\ v \neq 0 \in H_0^2(D):\ a_\lambda(v, \Psi) = 0 \quad \forall\ \Psi \in H_0^2(D)$$

where

$$a_\lambda(v, \Psi) := \iint\limits_D (\Delta + \lambda(1+q))v(\Delta + \lambda(1+p))\Psi\ \frac{dx}{q-p} \tag{3.1}$$

Proof. First let $\lambda > 0$ be an interior transmission eigenvalue.
In this case we define $v := w - u$ with w and u as in Definition 1.2.
We observe that $v \in H_0^2(D)$ and $v \neq 0$. To see this assume $v = 0$. Then $u = w$ in D and thus $\lambda qu = \lambda pu$ in the weak sense (Compare to 2. and 3. of Definition 1.2). But then $\lambda(q-p)u = 0$ is also valid in the weak sense, which means that fully

written we have

$$0 = \iint_D u \cdot \lambda(q-p)\Psi \; dx \quad \forall \; \Psi \in H_0^2(D)$$

From Lemma 2.53, $\lambda > 0$ and $q - p \geq q_0 > 0$ we can conclude $u = 0$. Then $w = v + u = 0$ which is a contradiction to u, w being nontrivial.

Now we can compute that

$$\begin{aligned}(\Delta + \lambda(1+q))v &= (\Delta + \lambda(1+q))(w-u) \\ &= (\Delta + \lambda(1+q))w - 0 \\ &= (\Delta + \lambda(1+p))w + \lambda(q-p)w \\ &= 0 + \lambda(q-p)w \\ &= \lambda(q-p)w\end{aligned}$$

is valid in the weak sense.
This means that fully written we have for all $\Psi \in H_0^2(D)$ with the help of Green's second formula (Theorem 2.55)

$$\begin{aligned}\iint_D w \cdot \lambda(q-p)\Psi \; dx &= \iint_D v \cdot (\Delta + \lambda(1+q))\Psi \; dx \\ &= \iint_D v \cdot \Delta\Psi \; dx + \iint_D v \cdot \lambda(1+q)\Psi \; dx \\ &= \iint_D \Delta v \cdot \Psi \; dx + \iint_D v\lambda(1+q) \cdot \Psi \; dx \\ &= \iint_D (\Delta + \lambda(1+q))v \cdot \Psi \; dx \\ \Longrightarrow \; 0 &= \iint_D (w\lambda(q-p) - (\Delta + \lambda(1+q))v)\Psi \; dx\end{aligned}$$

Using Lemma 2.53 again we can conclude

$$0 = w\lambda(q-p) - (\Delta + \lambda(1+q))v$$

in $L^2(D)$. And from here we directly see that

$$\begin{aligned}&\iint_D w\lambda(q-p) \cdot \frac{1}{q-p}(\Delta + \lambda(1+p))\Psi \; dx \\ =&\iint_D (\Delta + \lambda(1+q))v \cdot \frac{1}{q-p}(\Delta + \lambda(1+p))\Psi \; dx\end{aligned}$$

is valid for all Ψ in $H_0^2(D)$. The left hand side of this equation is zero by choice of w and the right hand side is equal to $a_\lambda(v,\Psi)$ by definition of a_λ. Thus the condition

$$a_\lambda(v,\Psi) = 0 \quad \forall\ \Psi \in H_0^2(D)$$

of the theorem is fulfilled.

Now let $\lambda > 0$ and let there be a $v \neq 0 \in H_0^2(D)$ such that $a_\lambda(v,\Psi) = 0$ for all $\Psi \in H_0^2(D)$. In this case we have to show that λ is an interior transmission eigenvalue. We set

$$w := \frac{1}{\lambda(q-p)}(\Delta + \lambda(1+q))v \tag{3.2}$$

and observe $w \in L^2(D)$ as well as

$$\begin{aligned} & 0 = a_\lambda(v,\Psi) \quad \forall\ \Psi \in H_0^2(D) \\ \Longrightarrow\ & 0 = \frac{1}{\lambda} a_\lambda(v,\Psi) = \iint\limits_D w(\Delta + \lambda(1+p))\Psi\ dx \quad \forall\ \Psi \in H_0^2(D) \end{aligned} \tag{3.3}$$

or $(\Delta + \lambda(1+p))w = 0$ in the weak sense.
Defining $u := w - v \in L^2(D)$ we see $u - w = -v \in H_0^2(D)$ and utilizing (3.3), (3.2) as well as Theorem 2.55 we can calculate

$$\begin{aligned} & \iint\limits_D u(\Delta + \lambda(1+q))\Psi\ dx \\ = & \iint\limits_D (w - v)(\Delta + \lambda(1+q))\Psi\ dx \\ = & \iint\limits_D w\lambda(q-p)\Psi + w(\Delta + \lambda(1+p))\Psi - v(\Delta + \lambda(1+q))\Psi\ dx \\ = & \iint\limits_D w\lambda(q-p)\Psi - v(\Delta + \lambda(1+q))\Psi\ dx \\ = & \iint\limits_D (\Delta + \lambda(1+q))v \cdot \Psi - v(\Delta + \lambda(1+q))\Psi\ dx \\ = & \iint\limits_D \Delta v \cdot \Psi - v\Delta\Psi\ dx \\ = & \ 0 \end{aligned}$$

for all $\Psi \in H_0^2(D)$. In other words now we also have $(\Delta + \lambda(1+q))u = 0$ in the weak sense.
$u - w = -v \in H_0^2(D)$ includes the boundary conditions of Definition 1.2 as discussed in the paragraph after Lemma 2.54.
Furthermore u, w are nontrivial because otherwise we would have $v = w - u = 0$ in contradiction to $v \neq 0$.
Hence λ satisfies all of the conditions of Definition 1.2 and thereby is an interior transmission eigenvalue. □

With Theorem 3.1 established we now continue our analysis along the lines of [9] to achieve the desired result of countability of the interior transmission eigenvalues. Our goal is to show that the interior transmission eigenvalues are also the eigenvalues of a compact operator and then use Theorem 2.22.

First we split up the bilinear form a_λ of Theorem 3.1

$$\begin{aligned} a_\lambda(v,\Psi) &= \iint\limits_D (\Delta + \lambda(1+q))v(\Delta + \lambda(1+p))\Psi \,\frac{dx}{q-p} \\ &= \iint\limits_D \Delta v \Delta \Psi \,\frac{dx}{q-p} \\ &+ \lambda \underbrace{\iint\limits_D (1+q)v\Delta\Psi + (1+p)\Psi\Delta v \,\frac{dx}{q-p}}_{=:b_1(v,\Psi)} \\ &+ \lambda^2 \underbrace{\iint\limits_D (1+q)(1+p)v\Psi \,\frac{dx}{q-p}}_{=:b_2(v,\Psi)} \\ &= a_0(v,\Psi) + \lambda b_1(v,\Psi) + \lambda^2 b_2(v,\Psi) \end{aligned} \tag{3.4}$$

We now want to apply our version of Riesz' representation theorem (Theorem 2.37) to b_1 and b_2. In order to do so we need to ensure that both b_1 and b_2 are linear and bounded in both variables.
Recall that in Theorem 2.57 we confirmed $H_0^2(D)$ with the norm $\langle .,. \rangle_{H_0^2(D)}$ to be a Hilbert space, this is the space that we use Theorem 2.37 on.
Tackling the boundedness we observe with the help of the Cauchy-Schwarz inequal-

ity (Lemma 2.4) for $L^2(D)$ and Lemma 2.58 as an additional tool

$$\begin{aligned}
|b_1(v,\Psi)| =& |\iint\limits_D \ (1+q)v\Delta\Psi + (1+p)\Psi\Delta v \ \frac{dx}{q-p}| \\
\leq& \iint\limits_D \ |1+q||v\Delta\Psi| + |1+p||\Psi\Delta v| \ \frac{dx}{q-p} \\
\leq& \ \|1+q\|_{L^\infty(D)} \iint\limits_D \ |v\Delta\Psi| \ \frac{dx}{q-p} + \|1+p\|_{L^\infty(D)} \iint\limits_D \ |\Psi\Delta v| \ \frac{dx}{q-p} \\
=& \ \|1+q\|_{L^\infty(D)} \iint\limits_D \ |\frac{v}{(q-p)^{\frac{1}{2}}}| \cdot |\frac{\Delta\Psi}{(q-p)^{\frac{1}{2}}}| \ dx \\
&+ \|1+p\|_{L^\infty(D)} \iint\limits_D \ |\frac{\Psi}{(q-p)^{\frac{1}{2}}}| \cdot |\frac{\Delta v}{(q-p)^{\frac{1}{2}}}| \ dx \\
\leq& \ \|1+q\|_{L^\infty(D)} (\iint\limits_D \ |\frac{v}{(q-p)^{\frac{1}{2}}}|^2 \ dx)^{\frac{1}{2}} \cdot (\iint\limits_D \ |\frac{\Delta\Psi}{(q-p)^{\frac{1}{2}}}|^2 \ dx)^{\frac{1}{2}} \\
&+ \|1+p\|_{L^\infty(D)} (\iint\limits_D \ |\frac{\Psi}{(q-p)^{\frac{1}{2}}}|^2 \ dx)^{\frac{1}{2}} \cdot (\iint\limits_D \ |\frac{\Delta v}{(q-p)^{\frac{1}{2}}}|^2 \ dx)^{\frac{1}{2}} \qquad (3.5) \\
\leq& \ \frac{1}{\sqrt{c}} \ \|1+q\|_{L^\infty(D)} \|v\|_{H_0^2(D)} \|\Psi\|_{H_0^2(D)} \\
&+ \frac{1}{\sqrt{c}} \ \|1+p\|_{L^\infty(D)} \|v\|_{H_0^2(D)} \|\Psi\|_{H_0^2(D)} \\
\leq& \ c_1 \ \|v\|_{H_0^2(D)} \|\Psi\|_{H_0^2(D)}
\end{aligned}$$

where c is the constant from Lemma 2.58 and $c_1 := \frac{1}{\sqrt{c}}(\|1+q\|_{L^\infty(D)} + \|1+p\|_{L^\infty(D)})$. Similarly

$$\begin{aligned}
|b_2(v,\Psi)| =& \ |\iint\limits_D \ (1+q)(1+p)v\Psi \ \frac{dx}{q-p}| \\
\leq& \ \underbrace{\|(1+q)(1+p)\|_{L^\infty(D)}}_{=:c_3} \iint\limits_D \ |\frac{v}{(q-p)^{\frac{1}{2}}}| \cdot |\frac{\Psi}{(q-p)^{\frac{1}{2}}}| \ dx \\
\leq& \ c_3 (\iint\limits_D \ |\frac{v}{(q-p)^{\frac{1}{2}}}|^2 \ dx)^{\frac{1}{2}} \cdot (\iint\limits_D \ |\frac{\Psi}{(q-p)^{\frac{1}{2}}}|^2 \ dx)^{\frac{1}{2}} \qquad (3.6) \\
\leq& \ \frac{1}{c} \ c_3 \ \|v\|_{H_0^2(D)} \|\Psi\|_{H_0^2(D)} \\
=& \ c_2 \ \|v\|_{H_0^2(D)} \|\Psi\|_{H_0^2(D)}
\end{aligned}$$

where $c_2 := \frac{1}{c} \cdot c_3$.

Linearity of the functionals passes on directly from the linearity of the Lebesgue integral and the linearity of the Laplace operator and can be easily seen without detailed calculation.

Hence Theorem 2.37 ensures that there exist bounded linear operators $B_1, B_2\colon H_0^2(D) \to H_0^2(D)$ such that

$$b_1(v,\Psi) = \langle B_1 v, \Psi\rangle_{H_0^2(D)} \quad \forall\, v,\Psi \in H_0^2(D) \tag{3.7}$$

$$b_2(v,\Psi) = \langle B_2 v, \Psi\rangle_{H_0^2(D)} \quad \forall\, v,\Psi \in H_0^2(D) \tag{3.8}$$

Utilizing this we are able to calculate an equivalent condition for that in Theorem 3.1.

$$\begin{aligned}
& 0 = a_\lambda(v,\Psi) && \forall\,\Psi \in H_0^2(D)\\
\iff\; & 0 = a_0(v,\Psi) + \lambda b_1(v,\Psi) + \lambda^2 b_2(v,\Psi) && \forall\,\Psi \in H_0^2(D)\\
\iff\; & 0 = \langle v,\Psi\rangle_{H_0^2(D)} + \lambda\langle B_1 v,\Psi\rangle_{H_0^2(D)} + \lambda^2\langle B_2 v,\Psi\rangle_{H_0^2(D)} && \forall\,\Psi \in H_0^2(D)\\
\iff\; & 0 = \langle v + \lambda B_1 v + \lambda^2 B_2 v, \Psi\rangle_{H_0^2(D)} && \forall\,\Psi \in H_0^2(D)\\
\iff\; & 0 = v + \lambda B_1 v + \lambda^2 B_2 v
\end{aligned}$$

Where in the last step $\Leftarrow$ is clear and for $\Rightarrow$ we can put $\Psi := v + \lambda B_1 v + \lambda^2 B_2 v$. In order not to get confused later, we summarize our progress so far.

Theorem 3.2. *$\lambda > 0$ is an interior transmission eigenvalue in the sense of Definition 1.2 if and only if there exists $v \in H_0^2(D)$, $v \neq 0$ such that*

$$0 = v + \lambda B_1 v + \lambda^2 B_2 \tag{3.9}$$

where B_1 and B_2 are the operators with 3.7 and 3.8 as defined above.

Lemma 3.3. *Both B_1 and B_2 are self-adjoint.*

Proof. We calculate that b_1 is symmetric.

$$\begin{aligned}
b_1(v,\Psi) - b_1(\Psi,v) &= \iint_D (1+q)v\Delta\Psi + (1+p)\Psi\Delta v \;\frac{dx}{q-p}\\
&\quad - \iint_D (1+q)\Psi\Delta v + (1+p)v\Delta\Psi \;\frac{dx}{q-p}\\
&= \iint_D (1+q-(1+p))v\Delta\Psi + (1+p-(1+q))\Psi\Delta v \;\frac{dx}{q-p}\\
&= \iint_D v\Delta\Psi\; dx - \iint_D \Psi\Delta v\; dx\\
&= 0
\end{aligned}$$

is true for any $v, \Psi \in H_0^2(D)$ by Theorem 2.55. As b_2 is obviously symmetric, we see from 3.7 and 3.8

$$\langle B_i v, \Psi\rangle_{H_0^2(D)} = b_i(v, \Psi) = b_i(\Psi, v) = \langle B_i \Psi, v\rangle_{H_0^2(D)} = \langle v, B_i \Psi\rangle_{H_0^2(D)}$$

for all $v, \Psi \in H_0^2(D)$ and $i \in \{1, 2\}$. This means that B_1 and B_2 are self-adjoint. □

Furthermore B_2 is nonnegative due to $q, p, q - p > 0$.

$$\langle B_2 v, v\rangle_{H_0^2(D)} = b_2(v, v) = \iint_D (1 + q)(1 + p)v^2 \,\frac{dx}{q - p} \geq 0 \quad \forall\, v \in H_0^2(D)$$

Now Lemma 2.14 ensures that there exists an Operator $B_2^{\frac{1}{2}} \in \mathcal{B}^+[H]$ such that $B_2^{\frac{1}{2}} B_2^{\frac{1}{2}} = B_2$. We define $z := \lambda B_2^{\frac{1}{2}} v$ and observe

$$\begin{aligned}
(3.9) \iff 0 &= z - \lambda B_2^{\frac{1}{2}} v,\\
0 &= v + \lambda B_1 v + \lambda B_2^{\frac{1}{2}} z\\
\iff \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} v \\ z \end{pmatrix} + \lambda \underbrace{\begin{pmatrix} B_1 & B_2^{\frac{1}{2}} \\ -B_2^{\frac{1}{2}} & 0 \end{pmatrix}}_{=:B} \begin{pmatrix} v \\ z \end{pmatrix} \qquad (3.10)\\
\iff B \begin{pmatrix} v \\ z \end{pmatrix} &= -\frac{1}{\lambda} \begin{pmatrix} v \\ z \end{pmatrix}
\end{aligned}$$

Hence for every interior transmission eigenvalue $\lambda > 0$ via Theorem 3.2 and the above calculations $-\frac{1}{\lambda}$ is an eigenvalue of the matrix operator B.

Looking at it from the other direction this means that the only candidates for interior transmission eigenvalues are of the form $-\frac{1}{\tau}$ where $\tau < 0$ has to be an eigenvalue of B.

Let conversely $\tau < 0$ be an eigenvalue of B with eigenvector $(x, y)^T \neq 0$. Then we see from the definition of B

$$-B_2^{\frac{1}{2}} x = \tau y$$

and hence $x \neq 0$. Were $x = 0$ then $y = 0$ in contradiction to $(x, y)^T \neq 0$. Furthermore

$$\begin{aligned}
&B_1 x + B_2^{\frac{1}{2}} y = \tau x\\
&\implies 0 = x - \frac{1}{\tau} B_1 x - \frac{1}{\tau} B_2^{\frac{1}{2}} y = x - \frac{1}{\tau} B_1 x + (-\frac{1}{\tau})^2 B_2 x
\end{aligned}$$

By Theorem 3.2 we see that $-\frac{1}{\tau}$ is an interior transmission eigenvalue because τ is negative.

Putting all this together we get the following theorem.

Theorem 3.4. *The set of interior transmission eigenvalues in the sense of Definition 1.2 is equal to the set* $\{-\frac{1}{\tau} \mid \tau < 0$ *is an eigenvalue of* $B\}$ *with the matrix operator*

$$B\colon H_0^2(D) \times H_0^2(D) \to H_0^2(D) \times H_0^2(D)$$

as defined in (3.10).

3.2 Deduction

Since we now have Theorem 3.4, we want to learn a little bit more about the spectrum of B. As mentioned before, the way we intend to do that is to show that B is a compact operator and then apply Theorem 2.22. First we show that the matrix entries of B are compact.

Lemma 3.5. *Both* B_1 *and* B_2 *are compact with respect to* $\|.\|_{H_0^2(D)}$.

Proof. Similarly to [10, Section 4.5] we use the characterisation of compactness given in Lemma 2.19.
Let $(x_n)_{n\in\mathbb{N}} \subset H_0^2(D)$ be a sequence that converges weakly to zero.
Utilizing (3.7), our earlier calculation (3.5) and $q - p \geq q_0$ we can conclude

$$\begin{aligned}
\|B_1 x_n\|^2_{H_0^2(D)} &= \langle B_1 x_n, B_1 x_n\rangle_{H_0^2(D)} = b_1(x_n, B_1 x_n) \\
&\leq \|1+q\|_{L^\infty(D)} (\iint_D |\frac{x_n}{(q-p)^{\frac{1}{2}}}|^2\, dx)^{\frac{1}{2}} \cdot (\iint_D |\frac{\Delta B_1 x_n}{(q-p)^{\frac{1}{2}}}|^2\, dx)^{\frac{1}{2}} \\
&\quad + \|1+p\|_{L^\infty(D)} (\iint_D |\frac{B_1 x_n}{(q-p)^{\frac{1}{2}}}|^2\, dx)^{\frac{1}{2}} \cdot (\iint_D |\frac{\Delta x_n}{(q-p)^{\frac{1}{2}}}|^2\, dx)^{\frac{1}{2}} \\
&\leq \|1+q\|_{L^\infty(D)} \frac{1}{\sqrt{q_0}} \|x_n\|_{L^2(D)} \|B_1 x_n\|_{H_0^2(D)} \\
&\quad + \|1+p\|_{L^\infty(D)} \frac{1}{\sqrt{q_0}} \|B_1 x_n\|_{L^2(D)} \|x_n\|_{H_0^2(D)}
\end{aligned} \tag{3.11}$$

$(\|x_n\|_{H_0^2(D)})_{n\in\mathbb{N}}$ is bounded by Lemma 2.18 because $(x_n)_{n\in\mathbb{N}}$ is a weakly convergent series in $H_0^2(D)$ and that means $(\|B_1 x_n\|_{H_0^2(D)})_{n\in\mathbb{N}}$ is bounded as well since B_1 is a bounded operator on $H_0^2(D)$.

Besides $x_n \overset{w}{\to} 0$ in $H_0^2(D)$ we also have $B_1 x_n \overset{w}{\to} 0$ in $H_0^2(D)$. To see this let $f \in H_0^2(D)^*$.

$$\begin{aligned}
f B_1(\alpha x + y) &= f(B_1(\alpha x + y)) = f(\alpha B_1 x + B_1 y) \\
&= \alpha f(B_1 x) + f(B_1 y) = \alpha f B_1(x) + f B_1(y)
\end{aligned}$$

is true for all $\alpha \in \mathbb{R}$ and for all $x, y \in H_0^2(D)$. Hence $fB_1 \in H_0^2(D)^*$ and thus $f(B_1 x_n) = fB_1(x_n) \to 0$ because $x_n \overset{w}{\to} 0$ in $H_0^2(D)$. As $f \in H_0^2(D)^*$ was chosen arbitrarily we have $B_1 x_n \overset{w}{\to} 0$ in $H_0^2(D)$.

With both $x_n \overset{w}{\to} 0$ and $B_1 x_n \overset{w}{\to} 0$ in $H_0^2(D)$ we can finally conclude that $\|x_n\|_{L^2(D)} \to 0$ and $\|B_1 x_n\|_{L^2(D)} \to 0$ by $H_0^2(D) \overset{c}{\hookrightarrow} L^2(D)$ (Lemma 2.64) and Lemma 2.65.

Hence (3.11) implies $\|B_1 x_n\|^2_{H_0^2(D)} \to 0$ from where we can conclude that B_1 is compact by Lemma 2.19.

For the compactness of B_2 let $(x_n)_{n\in\mathbb{N}} \subset H_0^2(D)$ such that $x_n \overset{w}{\to} 0$ yet again. Apply the same argument as above to see that $B_2 x_n \overset{w}{\to} 0$ in $H_0^2(D)$. Recalling our earlier calculation (3.6) as well as (3.8) we have similar to (3.11)

$$\begin{aligned} \|B_2 x_n\|^2_{H_0^2(D)} &= \langle B_2 x_n, B_2 x_n \rangle_{H_0^2(D)} = b_2(x_n, B_2 x_n) \\ &\leq \frac{c_3}{q_0} \|x_n\|_{L^2(D)} \|B_2 x_n\|_{L^2(D)} \end{aligned}$$

Once more utilizing $H_0^2(D) \overset{c}{\hookrightarrow} L^2(D)$ and Lemma 2.65 we see $\|B_2 x_n\|_{L^2(D)} \to 0$ as well as $\|x_n\|_{L^2(D)} \to 0$ and thus $\|B_2 x_n\|_{H_0^2(D)} \to 0$. As above Lemma 2.19 gives us the compactness of B_2 from here. □

Lemma 3.6. *$B_2^{\frac{1}{2}}$ and thus also $-B_2^{\frac{1}{2}}$ are compact.*

Proof. As we have already proven all the prerequisites, this is a simple application of Lemma 2.20. □

Because 0 is obviously compact with respect to any norm as it maps every sequence onto the constant sequence $(0)_{n\in\mathbb{N}}$, all the matrix entries of B are compact with respect to $\|.\|_{H_0^2(D)}$.

Since B is an operator from $H_0^2(D) \times H_0^2(D)$ onto itself we need to equip that space with an inner product. Similar to the situation in Euclidean geometry we can define an inner product on $H_0^2(D) \times H_0^2(D)$ through

$$\langle (a, b), (c, d) \rangle_{H_0^2(D)\times H_0^2(D)} := \langle a, c \rangle_{H_0^2(D)} + \langle b, d \rangle_{H_0^2(D)}$$

for all $(a, b), (c, d) \in H_0^2(D) \times H_0^2(D)$. That $\langle ., . \rangle_{H_0^2(D)\times H_0^2(D)}$ actually satisfies the conditions to be an inner product (see Definition 2.2) can be seen by some some very straightforward calculations or, as we will do it, by looking at it very sharply. $H_0^2(D) \times H_0^2(D)$ is a Hilbert space, since if $((a_n, b_n))_{n\in\mathbb{N}} \subset H_0^2(D) \times H_0^2(D)$ is a Cauchy sequence with respect to the norm

$$\|(a, b)\|_{H_0^2(D)\times H_0^2(D)} = \sqrt{\|a\|^2_{H_0^2(D)} + \|b\|^2_{H_0^2(D)}}$$

this implies that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $H_0^2(D)$ respectively. Then $\|a_n - a_\infty\|_{H_0^2(D)} \to 0$ and $\|b_n - b_\infty\|_{H_0^2(D)} \to 0$ for some $a_\infty, b_\infty \in H_0^2(D)$ due to the completeness of $H_0^2(D)$ and from that $\|(a_n, b_n) - (a_\infty, b_\infty)\|_{H_0^2(D)\times H_0^2(D)} \to 0$ is obvious. Hence every Cauchy sequence converges in $H_0^2(D) \times H_0^2(D)$.

Lemma 3.7. *B as defined in (3.10) is a compact linear operator from $H_0^2(D) \times H_0^2(D)$ onto itself.*

Proof. The linearity of B comes from the linearity of all of it's entries and can be seen, again, by looking sharply.

$$B = \begin{pmatrix} B_1 & B_2^{\frac{1}{2}} \\ -B_2^{\frac{1}{2}} & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}}_{=:M_1} + \underbrace{\begin{pmatrix} 0 & B_2^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}}_{=:M_2} + \underbrace{\begin{pmatrix} 0 & 0 \\ -B_2^{\frac{1}{2}} & 0 \end{pmatrix}}_{=:M_3} \tag{3.12}$$

We have already shown that B_1 as well as $B_2^{\frac{1}{2}}$ and $-B_2^{\frac{1}{2}}$ are compact (Lemma 3.5 and Lemma 3.6). Using this we can prove that all summands on the right hand side of (3.12) are compact.

Let $((a_n, b_n))_{n\in\mathbb{N}} \subset H_0^2(D) \times H_0^2(D)$ be a bounded sequence. Then both $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded sequences in $H_0^2(D)$.

Because B_1 is compact there exists a subsequence $(B_1 a_{n_k})_{k\in\mathbb{N}}$ of $(B_1 a_n)_{n\in\mathbb{N}}$ and an $a_\infty \in H_0^2(D)$ such that $B_1 a_{n_k} \to a_\infty$ as k goes to infinity.

$$\| \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_{n_k} \\ b_{n_k} \end{pmatrix} - \begin{pmatrix} a_\infty \\ 0 \end{pmatrix} \|^2_{H_0^2(D)\times H_0^2(D)} = \|B_1 a_{n_k} - a_\infty\|^2_{H_0^2(D)}$$

The right hand side goes to zero as k goes to infinity, so we have found a convergent subsequence of $M_1(a_n, b_n)$ and hence M_1 is compact. Compactness of M_2 and M_3 follows by the exact same argument and from here the compactness of B follows by Lemma 2.21. □

Since we have now shown that B is a compact operator on a Hilbert space, the conclusions of Theorem 2.22 about it's spectrum hold true. In particular the set of eigenvalues of B is countable and has zero as only possible accumulation point. Combine this with Theorem 3.4 and we can summarize all of our work in this chapter.

Theorem 3.8. *Under the restrictions of Definition 1.1 the interior transmission eigenvalues in the sense of Definition 1.2 are countable. Their only possible accumulation point is infinity.*

Chapter 4

Existence

Having one of the two main goals of this thesis achieved with Theorem 3.8, we now want to show the existence of a real positive interior transmission eigenvalue in the sense of Definition 1.2 under certain conditions. Similarly to chapter 3 we base our analysis on literature covering the case $p = 0$, namely [3]. We will do things slightly different this time though, because we are not going to modify the analysis of [3]. Instead we will apply the result from there in a weakened form to a suitable setting, repeat the idea of [3] in that setting and then carry on along the lines of [9].

Theorem 4.1. *Let $F \subseteq \mathbb{R}^2$ be an open disc and let $q \in L^\infty(F)$ satisfy one of the following two conditions:*

a) $q \geq q_0$ *for some* $q_0 > 0$ *almost everywhere in* F

b) $-1 + q_1 \leq q \leq -q_2$ *for some* $q_1, q_2 > 0$ *almost everywhere in* F

Then there exist $\lambda > 0$ and nontrivial $u, w \in L^2(F)$ such that

1. $u - w \in H_0^2(F)$
2. $\Delta u + \lambda(1+q)u = 0 \quad$ *in* F
3. $\Delta w + \lambda w = 0 \quad$ *in* F
4. $u = w$ *and* $\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu}$ *on the boundary* ∂F *of* F

where 2. *and* 3. *have to be understood in the weak sense, i.e.*

$$0 = \iint_F u \cdot (\Delta + \lambda(1+q))\Psi \, dx \qquad \forall \, \Psi \in H_0^2(F)$$

$$0 = \iint_F w \cdot (\Delta + \lambda)\Psi \, dx \qquad \forall \, \Psi \in H_0^2(F)$$

Theorem 4.1 is the aforementioned weakened version of [3, Theorem 3.1]. Weakened because [3] assumes a more general domain than we do and [3] also allows for $q_1 = 0$ in condition b). By demanding $q_1 > 0$ we accept a slight loss of generality compared to [3], but gain a huge advantage that will become apparent in section 4.3. Also, since in our setting Theorem 4.1 is mostly a tool to prove the main theorem of this chapter, Theorem 4.7, the loss of generality does not concern us much.

We postpone proving Theorem 4.1 until the end of this chapter and immediately start the discussion how it can be translated into our setting.

Following [9] we define $A_\lambda \colon H_0^2(D) \to H_0^2(D)$ for arbitrary $\lambda > 0$ through

$$A_\lambda := I + \lambda B_1 + \lambda^2 B_2 \tag{4.1}$$

where $I \colon H_0^2(D) \to H_0^2(D)$ is the identity mapping and B_1 and B_2 are the same as in chapter 3, recall (3.7) and (3.8). Furthermore a_λ is again the form defined through (3.1). Recalling the calculations that lead to Theorem 3.2 we see

$$\begin{aligned} \langle A_\lambda v, \Psi\rangle_{H_0^2(D)} &= \langle (I + \lambda B_1 + \lambda^2 B_2)v, \Psi\rangle_{H_0^2(D)} \\ &= \langle Iv, \Psi\rangle_{H_0^2(D)} + \langle \lambda B_1 v, \Psi\rangle_{H_0^2(D)} + \langle \lambda^2 B_2 v, \Psi\rangle_{H_0^2(D)} \\ &= a_0(v,\Psi) + \lambda b_1(v,\Psi) + \lambda^2 b_2(v,\Psi) \\ &= a_\lambda(v,\Psi) \end{aligned} \tag{4.2}$$

for all v, Ψ in $H_0^2(D)$. Similarly to [9] we intend to find a $\tau > 0$ and a $v \in H_0^2(D)$ such that $a_\tau(v,v) < 0$. This would imply $\langle A_\tau v, v\rangle_{H_0^2(D)} < 0$ and going from there will be able to conclude that there exists a $0 < \lambda < \tau$ such that A_λ has zero as an eigenvalue. Via theorem 3.2 such a λ is then an interior transmission eigenvalue.

4.1 Construction

Let us search for said τ and v first. Recall from Definition 1.1 that we have

$$q \geq p + q_0 \geq q_0 > 0$$

almost everywhere in D. Choose an open disc $F \subseteq D$, $\epsilon_p > 0$ and define

$$p_0 := \|\tilde{p}\|_{L^\infty(F)} + \epsilon_p$$

where $\tilde{p}$ denotes the restriction of p on F.
What we also need is the following $\tilde{q} \in L^\infty(F)$ for which we immediately calculate an inequality.

$$\tilde{q} := \frac{1+\tilde{p}}{1+p_0} - 1 \leq \frac{1+p_0-\epsilon_p}{1+p_0} - 1 = -\frac{\epsilon_p}{1+p_0} < 0 \quad \textit{almost everywhere in } F$$

Furthermore

$$-1 + \frac{1}{1+p_0} \leq -1 + \frac{1+\tilde{p}}{1+p_0} = \tilde{q} \quad \textit{almost everywhere in } F$$

We observe that $\tilde{q}$ satisfies condition b) of Theorem 4.1 applied to F. Theorem 4.1 gives us a $\lambda > 0$ and nontrivial $\tilde{u},\ \tilde{w}\ \in L^2(F)$ such that

(i) $\tilde{u} - \tilde{w} \in H_0^2(F)$

(ii) $\Delta\tilde{u} + \lambda(1+\tilde{q})\tilde{u} = 0 \quad \textit{in } F$

(iii) $\Delta\tilde{w} + \lambda\tilde{w} = 0 \quad \textit{in } F$

with (ii) and (iii) in the weak sense. Plugging in the definition of $\tilde{q}$ we have

$$\text{(ii)} \iff \Delta\tilde{u} + \lambda(\frac{1+\tilde{p}}{1+p_0})\tilde{u} = 0 \quad \text{in } F$$

We set $\tilde{v} := \tilde{u} - \tilde{w}$, extend u and w to all of D by zero

$$u := \begin{cases} \tilde{u} & \textit{in } F \\ 0 & \textit{in } D \setminus F \end{cases} \qquad\qquad w := \begin{cases} \tilde{w} & \textit{in } F \\ 0 & \textit{in } D \setminus F \end{cases}$$

and realize that $u, w \in L^2(D)$ because of

$$\iint\limits_D |u|^2\ dx = \iint\limits_F |u|^2\ dx + \iint\limits_{D\setminus F} |u|^2\ dx = \iint\limits_F |u|^2\ dx$$

and the same calculation for w.

Lemma 4.2. $v := u - w \in H_0^2(D)$

Proof. (i) says that $\tilde{v} := \tilde{u} - \tilde{w} \in H_0^2(F)$. Hence by Definition 2.51 there must exist a sequence $(\tilde{\varphi}_n) \subset C_0^\infty(F)$ that converges to $\tilde{v}$ in the $\|.\|_2$ norm on F. Extending the $\tilde{\varphi}_n$ to all of D by zero

$$\varphi_n := \begin{cases} \tilde{\varphi}_n & \textit{in } F \\ 0 & \textit{in } D \setminus F \end{cases}$$

for all $n \in \mathbb{N}$ we have $(\varphi_n) \subset C_0^\infty(D)$. This is because outside of $\overline{F}$ $\varphi_n = 0$ is clearly C^∞, inside of F $\varphi_n = \tilde{\varphi}_n$ is C^∞ by choice of $\tilde{\varphi}_n$ and for x on the boundary ∂F of our open disc F we can find some $\epsilon > 0$ such that the disc $K(x,\epsilon)$ of radius ϵ centered around x is disjoint to the compact support of $\tilde{\varphi}_n$. Then $\varphi_n = 0$ in $K(x,\epsilon)$ and hence φ_n is C^∞ in x.

Because both v and φ_n are equal to zero in $D\setminus F$, we see from Definition 2.46 that the norm $\|v - \varphi_n\|_2$ in D is equal to the norm $\|\tilde{v} - \tilde{\varphi}_n\|_2$ in F.

But as $\tilde{\varphi}_n \to \tilde{v}$ with respect to $\|.\|_2$ in F that means $\varphi_n \to v$ with respect to $\|.\|_2$ in D. Hence $v \in H_0^2(D)$. □

Setting $\tau := \frac{\lambda}{1+p_0}$ we have $\lambda = \tau(1+p_0)$ and see (ii) $\iff$ (v), (iii) $\iff$ (vi) with

(v) $\Delta\tilde{u} + \tau(1+\tilde{p})\tilde{u} = 0 \quad in\ F$

(vi) $\Delta\tilde{w} + \tau(1+p_0)\tilde{w} = 0 \quad in\ F$

Now almost everything is in place for our calculation to show that $a_\tau(v,v) < 0$, the only thing missing is the restriction $q \geq p_0$ in F. We assume that is valid for the moment and observe

$$
\begin{aligned}
a_\tau(v,v) &= \iint\limits_D (\Delta + \tau(1+q))v(\Delta + \tau(1+p))v \, \frac{dx}{q-p} \\
&= \iint\limits_F (\Delta + \tau(1+q))v(\Delta + \tau(1+p))v \, \frac{dx}{q-p} \\
&= \iint\limits_F (\Delta + \tau(1+p+(q-p)))v(\Delta + \tau(1+p))v \, \frac{dx}{q-p} \\
&= \iint\limits_F ((\Delta + \tau(1+p))v)^2 + \tau(q-p)v(\Delta + \tau(1+p))v \, \frac{dx}{q-p} \\
&= \iint\limits_F ((\Delta + \tau(1+p))v)^2 \, \frac{dx}{q-p} + \iint\limits_F \tau v(\Delta + \tau(1+p))v \, dx \\
&\leq \iint\limits_F ((\Delta + \tau(1+p))v)^2 \, \frac{dx}{p_0-p} + \iint\limits_F \tau v(\Delta + \tau(1+p))v \, dx \\
&= \iint\limits_F (\Delta + \tau(1+p_0))v(\Delta + \tau(1+p))v \, \frac{dx}{p_0-p} \\
&\quad + \iint\limits_F \tau(p-p_0)v(\Delta + \tau(1+p))v \, \frac{dx}{p_0-p} + \iint\limits_F \tau v(\Delta + \tau(1+p))v \, dx \\
&= \iint\limits_F (\Delta + \tau(1+p_0))v(\Delta + \tau(1+p))v \, \frac{dx}{p_0-p} \\
&= \iint\limits_F (\Delta + \tau(1+p_0))\tilde{v}(\Delta + \tau(1+p))\tilde{v} \, \frac{dx}{p_0-\tilde{p}} \\
&= a_\tau^F(\tilde{v},\tilde{v})
\end{aligned} \tag{4.3}
$$

where a_τ^F is the bilinear form corresponding to Theorem 3.1 applied to the problem in F with scattering object $q^F = p_0$ and background medium $p^F = \tilde{p}$.

Note that $p_0 \geq \tilde{p}(x) + \epsilon_p \geq \epsilon_p > 0$ for almost all $x \in F$ and recall that we have $\tau > 0$ and $\tilde{u}, \tilde{w} \in L^2(F)$ with the properties (i), (vi) and (v). Compare this to our definitions 1.1 and 1.2 and observe that τ is indeed an interior transmission eigenvalue for this new problem with $p_0, \tilde{p}, \epsilon_p, F, \tau, \tilde{u}, \tilde{w}$ corresponding to the variables $q, p, q_0, D, \lambda, u, w$ of our definitions 1.1 and 1.2 in that order.

In the proof of Theorem 3.1 we showed that by defining $v := w - u$ we get $a_\lambda(v, \Psi) = 0$ for all $\Psi \in H_0^2(D)$ for our usual notation provided λ is an interior transmission eigenvalue. Through mere renaming we can thus conclude that for $\tilde{v} = \tilde{u} - \tilde{w}$ we have $a_\tau^F(\tilde{v}, \Psi) = 0$ for all $\Psi \in H_0^2(F)$ and in particular $a_\tau^F(\tilde{v}, \tilde{v}) = 0$ because $\tilde{v} \in H_0^2(F)$. With this knowledge (4.3) becomes

$$a_\tau(v, v) \leq 0 \tag{4.4}$$

As said before the calculation for (4.3) we needed $q \geq p_0$ in F here. Now we profit from our indecisive choice of F and ϵ_p. Recall that $p_0 = \|\tilde{p}\|_{L^\infty(F)} + \epsilon_p$ for a freely chosen $\epsilon_p > 0$. Since in Definition 1.1 we already assumed $p, q \in L^\infty(D)$ as well as $q(x) \geq p(x) + q_0$ for almost all $x \in D$ and a $q_0 > 0$, fulfilling the restriction $q \geq p_0$ for some disc $F \subseteq D$ and some $\epsilon_p > 0$ is basically always possible.

But as at this point we do not want to start an expedition into the land of measure theory, we simply demand piecewise continuity for q and p. The following lemma summarizes our progress thus far.

Lemma 4.3. *Assume additionally to the setting in Definition 1.1 that p and q are piecewise continuous, i.e. continuous everywhere in D except in a finite number of points. Then there exist an open disc $F \subseteq D$ and an $\epsilon_p > 0$ such that $q \geq p_0$ in F where $p_0 := \|\tilde{p}\|_{L^\infty(F)} + \epsilon_p$ and $\tilde{p}$ denotes the restriction of p onto F.*
In this case there exist $v \in H_0^2(D)$ and $\tau > 0$ such that $a_\tau(v, v) \leq 0$.

Proof. Because q is piecewise continuous we find an open disc $K_1 \subseteq D$ such that q is continuous in K_1. Due to p being piecewise continuous in D and thus also in K_1, we find an open disc $K_2 \subseteq K_1$ such that p is continuous in K_2. Choose any $x \in K_2$. Because q and p are continuous in K_2 we can find $r > 0$ such that both

$$|q(y) - q(x)| \leq \frac{q_0}{4} \quad \forall y \in K_2 \text{ such that } \|x - y\| \leq r$$
$$|p(y) - p(x)| \leq \frac{q_0}{4} \quad \forall y \in K_2 \text{ such that } \|x - y\| \leq r$$

where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^2$. Finally we choose an open disc $F \subseteq K_2$ centered around x with a radius not larger than r and $\epsilon_p := \frac{q_0}{2}$. Observe that for all $y, z \in F$

$$q(y) \geq q(x) - \frac{q_0}{4} \geq p(x) + q_0 - \frac{q_0}{4} \geq p(z) - \frac{q_0}{4} + q_0 - \frac{q_0}{4} = p(z) + \epsilon_p$$

and hence

$$q \geq \|\tilde{p}\|_{L^\infty(F)} + \epsilon_p$$

The second part of the lemma is then merely a summary what we have already proven in this section and can be seen from (4.4). □

4.2 Deduction

Now we need to conclude how $a_\tau(v,v) \leq 0$ leads to the existence of an interior transmission eigenvalue for our original problem. We will be following [9] closely. Throughout the section $\tau > 0$ will be the one from Lemma 4.3 while $\lambda > 0$ is variable. From (4.2) we have

$$\langle A_\tau v, v\rangle_{H_0^2(D)} = a_\tau(v,v) \leq 0 \tag{4.5}$$

where $A_\lambda = I + \lambda B_1 + \lambda^2 B_2$ as defined in (4.1). In Lemma 3.5 we have shown that both B_1 and B_2 are compact operators with respect to $\|.\|_{H_0^2(D)}$ which leads us to the conclusion that

$$C_\lambda := \lambda B_1 + \lambda^2 B_2$$

also is a compact operator for any $\lambda \in \mathbb{R}$ by Lemma 2.21.

Lemma 3.3 ensures that both A_λ and C_λ are self-adjoint as one can easily see from Definition 2.11 that any linear combination of self-adjoint operators is self-adjoint.

We use Lemma 2.34 to transport everything in the complex Hilbert space $\tilde{H}_0^2(D)$ and observe for any $\lambda \in \mathbb{R}$ that because A_λ and C_λ are self-adjoint, $\tilde{A}_\lambda$ and $\tilde{C}_\lambda$ are as well and that because C_λ is compact, so is $\tilde{C}_\lambda$.

Lemma 4.4. *For any $\lambda \in \mathbb{R}$ the spectrum of $\tilde{A}_\lambda$ consists of eigenvalues of $\tilde{A}_\lambda$ and possibly the 1 which is it's only possible accumulation point.*
In particular $\sigma(\tilde{A}_0) = \{1\}$

Proof. We apply Theorem 2.22 on $\tilde{C}_\lambda$ and see that $\sigma(\tilde{C}_\lambda)$ consists of eigenvalues of $\tilde{C}_\lambda$ and possibly the zero which is also it's only possible accumulation point.
By Definition 2.23, $\alpha \in \mathbb{C}$ is in the resolvent set $\rho(\tilde{A}_\lambda)$ if and only if $\alpha I - \tilde{A}_\lambda$ has a bounded inverse. However

$$\alpha I - \tilde{A}_\lambda = \alpha I - (I + \tilde{C}_\lambda) = (\alpha - 1)I - \tilde{C}_\lambda$$

implies that this the case if and only if $\alpha - 1 \in \rho(\tilde{C}_\lambda)$. Hence

$$\begin{aligned}\sigma(\tilde{A}_\lambda) &= \mathbb{C}\backslash\rho(\tilde{A}_\lambda) = \mathbb{C}\backslash\{\alpha \in \mathbb{C} \mid \alpha - 1 \in \rho(\tilde{C}_\lambda)\} \\ &= \{\alpha \in \mathbb{C} \mid \alpha - 1 \notin \rho(\tilde{C}_\lambda)\} = \{\alpha \in \mathbb{C} \mid \alpha - 1 \in \sigma(\tilde{C}_\lambda)\}\end{aligned}$$

which proves that 1 is the only possible accumulation point of $\sigma(\tilde{A}_\lambda)$.

Let $\alpha \in \sigma(\tilde{A}_\lambda)\backslash\{1\}$. Then $\alpha - 1 \neq 0$ is an eigenvalue of $\tilde{C}_\lambda$. Choose a corresponding eigenvector x and observe

$$\tilde{A}_\lambda x = (I + \tilde{C}_\lambda)x = Ix + \tilde{C}_\lambda x = x + (\alpha - 1)x = \alpha x$$

or in other words α is an eigenvalue of $\tilde{A}_\lambda$. Seeing that 1 is the only eigenvalue of $\tilde{A}_0 = I$ concludes the proof. □

Lemma 4.5. *A_τ and $\tilde{A}_\tau$ have a nonpositive real eigenvalue.*

Proof. We profit from $A_\tau = I + C_\tau$ where C_τ is compact and self-adjoint. Theorem 2.24 ensures that we can write

$$C_\tau = \sum_k \lambda_k \langle ., e_k\rangle_{H_0^2(D)} e_k$$

where the $\lambda_k \neq 0$ are eigenvalues of C and the corresponding eigenvectors e_k form an orthonormal system. This sum is not void because if C_τ were zero then from (4.5) $\langle v, v\rangle_{H_0^2(D)} \leq 0$ in contradiction to $v \neq 0$. Applying that characterization of C_τ to (4.5) we can calculate

$$\begin{aligned} 0 &\geq \langle A_\tau v, v\rangle_{H_0^2(D)} = \langle (I + C_\tau)v, v\rangle_{H_0^2(D)} = \langle v, v\rangle_{H_0^2(D)} + \langle C_\tau v, v\rangle_{H_0^2(D)} \\ &= \langle v, v\rangle_{H_0^2(D)} + \langle \sum_k \lambda_k \langle v, e_k\rangle_{H_0^2(D)} e_k, v\rangle_{H_0^2(D)} \\ &= \langle v, v\rangle_{H_0^2(D)} + \sum_k \lambda_k \langle v, e_k\rangle_{H_0^2(D)} \langle e_k, v\rangle_{H_0^2(D)} \end{aligned} \tag{4.6}$$

Assume $\lambda_k > -1$ for all k. In this case (4.6) becomes

$$\begin{aligned} 0 &\geq \langle v, v\rangle_{H_0^2(D)} + \sum_k \lambda_k \langle v, e_k\rangle_{H_0^2(D)} \langle e_k, v\rangle_{H_0^2(D)} \\ &> \langle v, v\rangle_{H_0^2(D)} + \sum_k -\langle v, e_k\rangle_{H_0^2(D)} \langle e_k, v\rangle_{H_0^2(D)} \end{aligned} \tag{4.7}$$

because $\langle v, e_k\rangle^2_{H_0^2(D)} \geq 0$ and at least one of the $\langle v, e_k\rangle_{H_0^2(D)}$ is not zero. Were they all zero, then $Cv = 0$ and hence $0 \geq \langle A_\tau v, v\rangle_{H_0^2(D)} = \langle v, v\rangle_{H_0^2(D)}$ in contradiction to $v \neq 0$. According to Lemma 2.25 we can write

$$v = y + \sum_k \langle v, e_k\rangle_{H_0^2(D)} e_k$$

for some y in the kernel of C_τ. Keep in mind that according to Lemma 2.25 y is orthogonal to $\sum_k \langle v, e_k\rangle_{H_0^2(D)} e_k$ and that the e_k form an orthonormal system.

$$\begin{aligned} \langle v, v\rangle_{H_0^2(D)} &= \langle y + \sum_k \langle v, e_k\rangle_{H_0^2(D)} e_k, y + \sum_i \langle v, e_i\rangle_{H_0^2(D)} e_i\rangle_{H_0^2(D)} \\ &= \langle y, y\rangle_{H_0^2(D)} + \langle \sum_k \langle v, e_k\rangle_{H_0^2(D)} e_k, \sum_i \langle v, e_i\rangle_{H_0^2(D)} e_i\rangle_{H_0^2(D)} \\ &\geq \sum_k \sum_i \langle v, e_i\rangle_{H_0^2(D)} \langle v, e_k\rangle_{H_0^2(D)} \langle e_k, e_i\rangle_{H_0^2(D)} \\ &= \sum_k \langle v, e_k\rangle_{H_0^2(D)} \langle v, e_k\rangle_{H_0^2(D)} \end{aligned} \tag{4.8}$$

Plugging (4.8) into (4.7) we get $0 > 0$ which is a contradiction.

Hence our assumption was wrong and there must be at least one $\tilde{k}$ such that $\lambda_{\tilde{k}} \leq -1$. Let $x \in H_0^2(D)$, $x \neq 0$ be an eigenvector of C_τ to the eigenvalue $\lambda_{\tilde{k}}$.

$$A_\tau x = (I + C_\tau)x = x + \lambda_{\tilde{k}} x = (1 + \lambda_{\tilde{k}})x$$

shows us that x is an eigenvector of A_τ to the eigenvalue $1 + \lambda_{\tilde{k}} \leq 0$.
Lemma 2.35 gives the assertion for $\tilde{A}_\tau$. □

Lemma 4.6. *There exists $\lambda \in \mathbb{R}$, $0 < \lambda \leq \tau$ such that $\tilde{A}_\lambda$ has zero as an eigenvalue.*

Proof. Let $\lambda \in \mathbb{R}$ be arbitrary.
$\emptyset \neq \sigma(\tilde{A}_\lambda) \subseteq \mathbb{R}$ due to Lemma 2.28 and Lemma 2.27 because $\tilde{A}_\lambda$ is bounded and self-adjoint. This combined with the knowledge that minus infinity is not a possible accumulation point of $\sigma(\tilde{A}_\lambda)$ (Lemma 4.4) ensures that the definition

$$t : [0, \tau] \to \mathbb{R}, \quad t(\lambda) = \inf\{\alpha \mid \alpha \in \sigma(\tilde{A}_\lambda)\} \tag{4.9}$$

makes sense. Let $\delta, \gamma \in [0, \tau]$. If $t(\delta) \leq t(\gamma)$ then by (4.9) we see

$$|t(\delta) - t(\gamma)| = dist(t(\delta), \sigma(\tilde{A}_\gamma)) \tag{4.10}$$

If $t(\gamma) \leq t(\delta)$ then analogously by (4.9)

$$|t(\gamma) - t(\delta)| = dist(t(\gamma), \sigma(\tilde{A}_\delta)) \tag{4.11}$$

Combining (4.10) and (4.11) we have for any $\delta, \gamma \in [0, \tau]$

$$|t(\gamma) - t(\delta)| \leq dist(t(\delta), \sigma(\tilde{A}_\gamma)) + dist(t(\gamma), \sigma(\tilde{A}_\delta)) \tag{4.12}$$

Looking at (4.9) again we see that either $t(\delta) \in \sigma(\tilde{A}_\delta)$ or there exists a sequence in $\sigma(\tilde{A}_\delta)$ that converges to $t(\delta)$. In either case we have

$$dist(t(\delta), \sigma(\tilde{A}_\gamma)) \leq \sup_{\alpha \in \sigma(\tilde{A}_\delta)} dist(\alpha, \sigma(\tilde{A}_\gamma)) \tag{4.13}$$

Applying the same argument for $t(\gamma)$ we get

$$dist(t(\gamma), \sigma(\tilde{A}_\delta)) \leq \sup_{\alpha \in \sigma(\tilde{A}_\gamma)} dist(\alpha, \sigma(\tilde{A}_\delta)) \tag{4.14}$$

Plugging (4.13) and (4.14) into (4.12) we have for any $\delta, \gamma \in [0, \tau]$

$$|t(\gamma) - t(\delta)| \leq \sup_{\alpha \in \sigma(\tilde{A}_\delta)} dist(\alpha, \sigma(\tilde{A}_\gamma)) + \sup_{\alpha \in \sigma(\tilde{A}_\gamma)} dist(\alpha, \sigma(\tilde{A}_\delta)) \tag{4.15}$$

Now observe

$$\begin{aligned}\tilde{A}_\delta - \tilde{A}_\gamma &= (1-1)I + \underbrace{(\delta-\gamma)\tilde{B}_1 + (\delta^2-\gamma^2)\tilde{B}_2}_{=:\tilde{C}_{\delta,\gamma}} \\ \implies \tilde{A}_\delta &= \tilde{A}_\gamma + \tilde{C}_{\delta,\gamma} \qquad (4.16)\end{aligned}$$

$\tilde{C}_{\delta,\gamma}$ is self-adjoint as a sum of self-adjoint operators. Apply Theorem 2.29 on (4.16) to see that for any $\delta, \gamma \in [0, \tau]$

$$\sup_{\alpha\in\sigma(\tilde{A}_\delta)} dist(\alpha, \sigma(\tilde{A}_\gamma)) + \sup_{\alpha\in\sigma(\tilde{A}_\gamma)} dist(\alpha, \sigma(\tilde{A}_\delta)) \le 2\|\tilde{C}_{\delta,\gamma}\| \qquad (4.17)$$

We plug (4.17) into (4.15) and calculate with the triangle inequality and multiplicativity for operator norms (Lemma 2.10)

$$\begin{aligned}|t(\gamma) - t(\delta)| \le 2\|\tilde{C}_{\delta,\gamma}\| &= \|(\delta-\gamma)\tilde{B}_1 + (\delta^2-\gamma^2)\tilde{B}_2\| \\ &\le |\delta-\gamma|\|\tilde{B}_1\| + |\delta^2-\gamma^2|\|\tilde{B}_2\| \qquad (4.18)\end{aligned}$$

The right hand side of (4.18) clearly converges to zero as $\gamma \to \delta$ which implies that the left hand side does the same. In other words t is continuous as a function from $[0, \tau]$ to $\mathbb{R}$.
From Lemma 4.4 we see $t(0) = 1$ while Lemma 4.5 ensures $t(\tau) \le 0$.
The classical intermediate value theorem of analysis in $\mathbb{R}^1$ [6, §11] then ensures that there exists $\lambda \in (0, \tau]$ such that $t(\lambda) = 0$. From (4.9) and Lemma 4.4 we see that 0 is in $\sigma(\tilde{A}_\lambda)$ and is indeed an eigenvalue of $\tilde{A}_\lambda$. □

By Lemma 2.35 we can convert the result of Lemma 4.6 back in the real case and now know that there exists $\lambda > 0$ such that A_λ has zero as an eigenvalue. After recalling the $A_\lambda = I + \lambda B_1 + \lambda^2 B_2$, Theorem 3.2 ensures that this λ is indeed an interior transmission eigenvalue.

Again we summarize our progress in this chapter so far.

Theorem 4.7. *Under the restrictions of Definition 1.1 and the assumptions of Lemma 4.3 there exists at least one interior transmission eigenvalues in the sense of Definition 1.2.*

What we are still missing for this to be completely proven however is the proof of Theorem 4.1 which we will do in our next section.

4.3 Supplement

Although Theorem 4.1 is already proven in [3], we want to give a proof of it here, because we are now in the comfortable position where we can recycle a lot of what we have done earlier. What we still have to do is basically simply to juggle some numbers.

Setting certain p and q in Definition 1.1 is effortlessly possible and hence we can use everything said in this thesis so far except section 4.1 (where we have used Theorem 4.1) for the case where those p and q fulfill the restrictions of Definition 1.1.

Recall that in section 4.2 we used the $v \in H_0^2(D)$ and $\tau > 0$ such that $a_\tau(v, v) \leq 0$ constructed in section 4.1 to prove the existence of an interior transmission eigenvalue.

Assume condition a) of Theorem 4.1 holds true for $\tilde{q} \in L^\infty(F)$. In this case we set $p = 0$ and $q = \tilde{q}$ as well as $D = F$ in Definition 1.1 and observe that the conditions are fulfilled. Thus if we are able to construct $v \in H_0^2(F)$ and $\tau > 0$ such that the corresponding $a_\tau(v, v) \leq 0$, then this completes the proof of Theorem 4.1. These will be constructed in Lemma 4.10.

Now assume condition b) of Theorem 4.1 holds true for $q \in L^\infty(F)$. We quickly recall it for convenience.

$$-1 + q_1 \leq q \leq -q_2 \textit{ almost everywhere in } F \textit{ for some } q_1, q_2 > 0$$

Sadly we realize that simply setting $p = 0$ in Definition 1.1 is not enough, because q does not satisfy the restriction of Definition 1.1. However we can help us with a trick. Because $q_1 > 0$ there exist $k \in \mathbb{N}$ such that $kq_1 \geq 1$. For any such k define

$$\tilde{p} := k - 1 + kq \qquad\qquad \tilde{q} := k - 1$$

and observe

$$\begin{aligned} \tilde{p} &= k - 1 + kq \geq k - 1 + k(-1 + q_1) = -1 + kq_1 \geq 0 \\ \tilde{q} &= k - 1 \geq k - 1 + k(q + q_2) = \tilde{p} + kq_2 \end{aligned}$$

Going back to Definition 1.1 we observe that $\tilde{q}$ and $\tilde{p}$ satisfy the restrictions with $q_0 := kq_2 > 0$.
Items 1. and 2. of Definition 1.2 become

$$\begin{aligned} \Delta u + \lambda(1 + (k-1))u = 0 &\iff \Delta u + \lambda k u = 0 \\ \Delta w + \lambda(1 + (k - 1 + kq))w = 0 &\iff \Delta w + \lambda k(1 + q)w = 0 \end{aligned}$$

But if there exists $\lambda > 0$ such that this is fulfilled, then $\frac{\lambda}{k} > 0$ fulfills everything Theorem 4.1 asks for. Hence analogously to the case of condition a) all we need to do is to construct $v \in H_0^2(F)$ and $\tau > 0$ such that the corresponding $a_\tau(v, v) \leq 0$ to conclude the proof. That will be done in Lemma 4.11.

Similar to [3] we will take the existence of an interior transmission eigenvalue for constant index of refraction $q > 0$ and a disc for granted, i.e.

Lemma 4.8. *Let $F \subseteq \mathbb{R}^2$ be an open disc and let $q > 0$ be a constant. Then there exist $\lambda > 0$ and nontrivial $u, w \in L^2(F)$ such that*

1. $u - w \in H_0^2(F)$
2. $\Delta u + \lambda(1+q)u = 0$ *in* F
3. $\Delta w + \lambda w = 0$ *in* F
4. $u = w$ *and* $\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu}$ *on the boundary* ∂F *of* F

where 2. and 3. are in the weak sense.

For more information on this see chapter 5.

Lemma 4.9. *Let $F \subseteq \mathbb{R}^2$ be an open disc and let $q > 0$ be a constant. Then there exist $\lambda > 0$ and $v \in H_0^2(F)$, $v \neq 0$ such that*

$$\iint_F (\Delta + \lambda(1+q))v(\Delta + \lambda)v \; dx = 0$$

Proof. There exists an interior transmission eigenvalue $\lambda > 0$ because of Lemma 4.8. Since $q > 0$, $p = 0$ and F satisfy the conditions of Definition 1.1, Theorem 3.1 is valid. Applying that theorem concludes the proof. □

Lemma 4.10. *Let $F \subseteq \mathbb{R}^2$ be an open disc and $q \in L^\infty(F)$ such that $q(x) \geq q_0$ for some $q_0 > 0$ and almost all x in F. Then there exist $v \in H_0^2(F)$ and $\lambda > 0$ such that*

$$\iint_F (\Delta + \lambda(1+q))v(\Delta + \lambda)v \; \frac{dx}{q} \leq 0$$

Proof.

$$\begin{aligned}
&\iint_F (\Delta + \lambda(1+q))v(\Delta + \lambda)v \; \frac{dx}{q} \\
=&\iint_F (\Delta + \lambda)v(\Delta + \lambda)v \; \frac{dx}{q} + \iint_F \lambda q v(\Delta + \lambda)v \; \frac{dx}{q} \\
\leq&\iint_F (\Delta + \lambda)v(\Delta + \lambda)v \; \frac{dx}{q_0} + \iint_F \lambda q_0 v(\Delta + \lambda)v \; \frac{dx}{q_0} \\
=&\frac{1}{q_0}\iint_F (\Delta + \lambda(1+q_0))v(\Delta + \lambda)v \; dx
\end{aligned} \tag{4.19}$$

By applying Lemma 4.9 we see that there exist $v \in H_0^2(F)$ and $\lambda > 0$ such that the right hand side of (4.19) is zero. □

Lemma 4.11. *Let $F \subseteq \mathbb{R}^2$ be an open disc. Let furthermore $q \in L^\infty(F)$ such that $-1 + q_1 \le q \le -q_2$ in F for $q_1, q_2 > 0$ almost everywhere in F. Choose $k \in \mathbb{N}$ such that $kq_1 \ge 1, kq_2 \ge 1$ and $k \ge 2$. Then there exist $v \in H_0^2(F)$ and $\lambda > 0$ such that*

$$\iint_F (\Delta + \lambda(1 + k - 1))v(\Delta + \lambda(1 + k - 1 + kq))v \, \frac{dx}{k - 1 - (k - 1 + kq)} \le 0$$

Proof.

$$\begin{aligned}
&\iint_F (\Delta + \lambda(1 + k - 1))v(\Delta + \lambda(1 + k - 1 + kq))v \, \frac{dx}{k - 1 - (k - 1 + kq)} \\
&= \iint_F (\Delta + \lambda k)v(\Delta + \lambda k(1 + q))v \, \frac{dx}{-kq} \\
&= \iint_F (\Delta + \lambda k)v(\Delta + \lambda k)v \, \frac{dx}{-kq} + \iint_F (\Delta + \lambda k)v\lambda kqv \, \frac{dx}{-kq} \\
&= \iint_F ((\Delta + \lambda k)v)^2 \, \frac{dx}{-kq} - \iint_F (\Delta + \lambda k)v\lambda v \, dx
\end{aligned} \tag{4.20}$$

and due to

$$q \le -q_2, \; kq_2 \ge 1 \implies kq \le -kq_2 \le -1 \implies -kq \ge 1 \implies \frac{1}{-kq} \le 1$$

we have, continuing our calculation (4.20),

$$\begin{aligned}
\cdots \quad &\le \iint_F ((\Delta + \lambda k)v)^2 \, dx - \iint_F (\Delta + \lambda k)v\lambda v \, dx \\
&= \iint_F (\Delta + \lambda k)v(\Delta + \lambda(k - 1))v \, dx
\end{aligned} \tag{4.21}$$

Applying Lemma 4.9 to the constant $\frac{k}{k-1} - 1 > 0$ we get $v \in H_0^2(F)$ and $\tau > 0$ such that

$$\begin{aligned}
0 &= \iint_F (\Delta + \tau(1 + \frac{k}{k - 1} - 1))v(\Delta + \tau)v \, dx \\
&= \iint_F (\Delta + \tau(\frac{k}{k - 1}))v(\Delta + \tau)v \, dx
\end{aligned}$$

By setting $\lambda := \frac{\tau}{k-1} > 0$ we see that the right hand side of (4.21) is zero for this v and λ. □

With these lemmata proven, the proof of Theorem 4.1 is complete and consequently also the proof of the main theorem of this chapter, Theorem 4.7.

Chapter 5

Open questions

This final chapter will cover some loose ends and provide an outlook on what further studies would be interesting. When it comes to the latter I will also include studies that would be interesting for me personally after writing this thesis, even if these subjects are well covered by literature already and thus might not be interesting from a researcher's point of view. As the name of the chapter might already suggest, we will slightly tone down on the formal aspects and use previously unintroduced notations and outside knowledge a little bit more liberally than before.

5.1 Constant index of refraction in a disc

The biggest question mark is clearly Lemma 4.8. When it first came up in chapter 4 we were following [3] which states that the lemma is valid. However the proof we are referred to [5, Theorem 2] is executed in $\mathbb{R}^3$ which warrants a closer examination. [5] restricts the search to spherically symmetric eigenfunctions and likewise we can restrict the search to radially symmetric eigenfunctions. We can not use the 'same' functions though, because the Laplacian in polar coordinates behaves slightly different than the Laplacian in spherical coordinates.

Unlike [5, Theorem 2] which covers any spherically stratified q, our Lemma 4.8 only is concerned with constant $q > 0$ which makes things easier. The idea behind our approach is still taken from [5] though.

Let $F \subseteq \mathbb{R}^2$ be an open disc of radius $a > 0$ and let $q > 0$ be a constant. The first thing we want to do is obviously to shift our coordinate system so that the origin is at the center of F. Having done that we search for strong solutions $u, w \in C^2(F)$ of the interior transmission problem that are radially symmetric, i.e. for all $(x_1, x_2) \in F$ we demand

$$u((x_1, x_2)) = u(\|(x_1, x_2)\|) \qquad w((x_1, x_2)) = w(\|(x_1, x_2)\|) \tag{5.1}$$

where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^2$. With solutions like this it makes a lot of sense to switch to polar coordinates, for details see [7, Section 206]. In short we have

$$x_1 = r\cos\varphi \qquad x_2 = r\sin\varphi \qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \tag{5.2}$$

and hence $\|(x_1, x_2)\| = r$. From the fact that F is a disc centered at the origin of our coordinate sytem we see that if we move along any vector orthogonal to it's boundary, we move only with respect to r in polar coordinates while φ remains fixed. This is important for the normal derivative in the boundary conditions. From the above, (5.1) and (5.2) we can conclude that the conditions of Lemma 4.8 are for any u, w fulfilling our demands equivalent to

$$\frac{\partial^2}{\partial r^2}u + \frac{1}{r}\frac{\partial}{\partial r}u + \lambda(1+q)u = 0 \quad \mathit{in}\ (0,a) \tag{5.3}$$

$$\frac{\partial^2}{\partial r^2}w + \frac{1}{r}\frac{\partial}{\partial r}w + \lambda w = 0 \quad \mathit{in}\ (0,a) \tag{5.4}$$

$$u(a) = w(a) \qquad \frac{\partial}{\partial r}u|_{r=a} = \frac{\partial}{\partial r}w|_{r=a} \tag{5.5}$$

Now we assume that u, w satisfy (5.3) - (5.5) and set

$$c := \sqrt{\lambda(1+q)} \qquad f(r) := u(\frac{r}{c}) \qquad g(r) := w(\frac{r}{\sqrt{\lambda}})$$

whereas f and g are defined only on $\mathbb{R}^1$. To avoid confusion we use the notation

$$\frac{\partial}{\partial r}u|_{r=y} =: u'(y) \qquad\qquad \frac{d}{dr}f|_{r=y} =: f'(y)$$

for both $u, w\colon F \to \mathbb{R}$ as well as for $f, g\colon \mathbb{R} \to \mathbb{R}$. f satisfies

$$\begin{aligned} y^2 f''(y) + yf'(y) + y^2 f(y) &= \frac{y^2}{c^2}u''(\frac{y}{c}) + \frac{y}{c}u'(\frac{y}{c}) + \frac{y^2}{c^2}c^2 u(\frac{y}{c}) \\ &= \frac{y^2}{c^2}(u''(\frac{y}{c}) + \frac{1}{\frac{y}{c}}u'(\frac{y}{c}) + \lambda(1+q)u(\frac{y}{c})) = 0 \end{aligned} \tag{5.6}$$

whenever $\frac{y}{c} \in (0,a)$ as one can see from (5.3).
Analogously g satisfies

$$\begin{aligned} y^2 g''(y) + yg'(y) + y^2 g(y) &= \frac{y^2}{\lambda}w''(\frac{y}{\sqrt{\lambda}}) + \frac{y}{\sqrt{\lambda}}u'(\frac{y}{\sqrt{\lambda}}) + \frac{y^2}{\lambda}\lambda u(\frac{y}{\sqrt{\lambda}}) \\ &= \frac{y^2}{\lambda}(u''(\frac{y}{\sqrt{\lambda}}) + \frac{1}{\frac{y}{\sqrt{\lambda}}}u'(\frac{y}{\sqrt{\lambda}}) + \lambda u(\frac{y}{\sqrt{\lambda}})) = 0 \end{aligned} \tag{5.7}$$

whenever $\frac{y}{\sqrt{\lambda}} \in (0,a)$.

(5.6) and (5.7) are a special case of what we call Bessel's differential equation. From [13, §25] we can conclude that their solutions must be a linear combination of the Bessel function of the first kind and order zero J_0 and the Bessel function of the second kind and order zero Y_0. However Y_0 has a singularity at zero and thus while it might solve (5.6) and (5.7), it can not lead us to a function in $C^2(F)$. Hence every solution of (5.6) or (5.7) that can possibly lead to a solution of (5.3) or (5.4) in $C^2(F)$ must be of the form

$$f(y) = c_1 J_0(y) \qquad\qquad g(y) = c_2 J_0(y) \tag{5.8}$$

where c_1 and c_2 are constants and J_0 is the Bessel function of the first kind and order zero defined through

$$J_0(y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} (\frac{y}{2})^{2k} \tag{5.9}$$

Because functions of the kind (5.8) solve (5.6) and (5.7) on all of $\mathbb{R}$, they lead to solutions of (5.3) and (5.4) by

$$u((x_1, x_2)) := c_1 J_0(c\|(x_1, x_2)\|) \qquad w((x_1, x_2)) := c_2 J_0(\sqrt{\lambda}\|(x_1, x_2)\|) \tag{5.10}$$

as quickly thinking through our chain of thought in reverse reveals. We observe that by above arguments (5.3) and (5.4) are fulfilled. Without going into details u and w are also in $C^2(F)$ because the power series in (5.9) converges for all $y \in \mathbb{R}$ [4, section 3.4]. Hence the system (5.3) - (5.5) has a solution with functions defined through (5.10) if and only if we can fulfill the boundary condition (5.5). That is the case if and only if

$$\begin{array}{rcl} c_1 J_0(ca) & = & c_2 J_0(\sqrt{\lambda} a) \\ c_1 c J_0'(ca) & = & c_2 \sqrt{\lambda} J_0'(\sqrt{\lambda} a) \end{array} \iff \underbrace{\begin{pmatrix} J_0(ca) & -J_0(\sqrt{\lambda} a) \\ c J_0'(ca) & -\sqrt{\lambda} J_0'(\sqrt{\lambda} a) \end{pmatrix}}_{=:M} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

is fulfilled. An equivalent conditions for the existence of nontrivial c_1, c_2 such that the above is true is that the matrix M does have determinant zero.

$$\det M = -J_0(ca)\sqrt{\lambda} J_0'(\sqrt{\lambda} a) + c J_0'(ca) J_0(\sqrt{\lambda} a) \tag{5.11}$$

Putting our progress in the form of a Lemma we get the following one.

Lemma 5.1. *Let $F \subseteq \mathbb{R}^2$ be an open disc of radius $a > 0$ and let $q > 0$ be a constant. Then the interior transmission eigenvalues $\lambda > 0$ with eigenfunctions of the form* (5.10) *are the zeros of* $\det M$ *as in* (5.11).

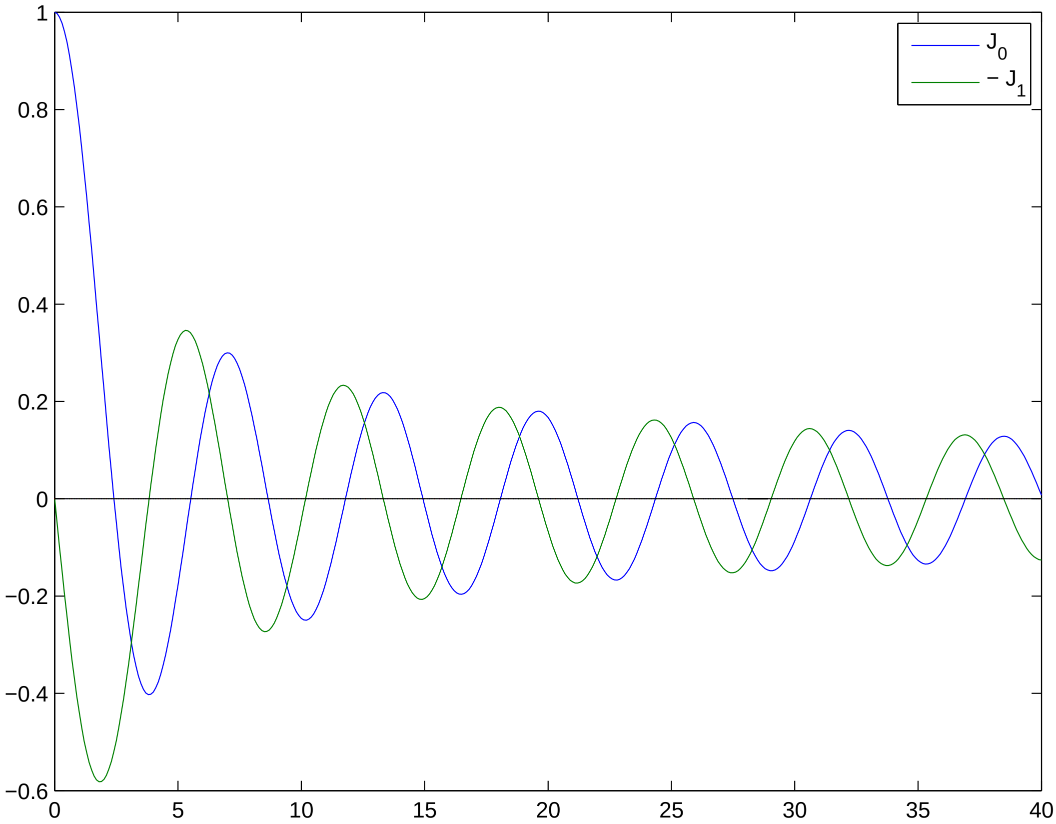

Figure 5.1: J_0 and it's derivative $J_0' = -J_1$

We can get a first idea why $\det M$ must have zeros from the behaviour of J_0 and J_0' as shown in Figure 5.1.

For a more profound analysis we need the following information on the asymptotical behaviour of these functions that can be deduced from [4, equation 3.59] and Euler's formula [6, §14].

$$
\begin{aligned}
J_0(y) &= \sqrt{\frac{2}{\pi y}}\cos(y - \frac{\pi}{4}) + \mathcal{O}(\frac{1}{y^{\frac{3}{2}}}) && as \quad y \to \infty \\
J_0'(y) &= \sqrt{\frac{2}{\pi y}}\cos(y + \frac{\pi}{4}) + \mathcal{O}(\frac{1}{y^{\frac{3}{2}}}) && as \quad y \to \infty
\end{aligned}
$$

Here $\mathcal{O}$ denotes the Landau symbol and means that the residual term is bounded by a constant times what's inside the '$\mathcal{O}$'. Recalling $c = \sqrt{\lambda(1+q)}$ and plugging the asymptotics above into (5.11) we get

$$
\begin{aligned}
\det M = &- \frac{2}{\pi a}\frac{1}{\sqrt[4]{1+q}}\cos(ca - \frac{\pi}{4})\cos(\sqrt{\lambda}a + \frac{\pi}{4}) \\
&+ \frac{2}{\pi a}\sqrt[4]{1+q}\cos(ca + \frac{\pi}{4})\cos(\sqrt{\lambda}a - \frac{\pi}{4}) + \mathcal{O}(\frac{1}{\sqrt{\lambda}})
\end{aligned} \tag{5.12}
$$

as $\lambda \to \infty$. From [6, §14] we can deduce the following equalities for all $x, y \in \mathbb{R}$.

$$
\begin{aligned}
\cos(x)\cos(y) &= \frac{1}{2}(\cos(x+y) + \cos(x-y)) \\
\cos(x - \frac{\pi}{2}) &= -\cos(x + \frac{\pi}{2})
\end{aligned}
$$

Using these on (5.12), for $\lambda \to \infty$ we can calculate the asymptotics

$$
\begin{aligned}
\pi a \det M = &- \frac{1}{\sqrt[4]{(1+q)}}(\cos(ca + \sqrt{\lambda}a) + \cos(ca - \sqrt{\lambda}a - \frac{\pi}{2})) \\
&+ \sqrt[4]{1+q}(\cos(ca + \sqrt{\lambda}a) + \cos(ca - \sqrt{\lambda}a + \frac{\pi}{2})) + \mathcal{O}(\frac{1}{\sqrt{\lambda}}) \\
= &\underbrace{\frac{\sqrt{1+q} - 1}{\sqrt[4]{1+q}}}_{=:A_1}\cos(ca + \sqrt{\lambda}a) + \underbrace{\frac{\sqrt{1+q} + 1}{\sqrt[4]{1+q}}}_{=:A_2}\cos(ca - \sqrt{\lambda}a + \frac{\pi}{2}) + \mathcal{O}(\frac{1}{\sqrt{\lambda}}) \\
= &A_1\cos(\sqrt{\lambda}\underbrace{a(\sqrt{1+q} + 1)}_{=:f_1}) + A_2\cos(\sqrt{\lambda}\underbrace{a(\sqrt{1+q} - 1)}_{=:f_2} + \frac{\pi}{2}) + \mathcal{O}(\frac{1}{\sqrt{\lambda}}) \\
= &A_1\cos(\sqrt{\lambda}f_1) + A_2\cos(\sqrt{\lambda}f_2 + \frac{\pi}{2}) + \mathcal{O}(\frac{1}{\sqrt{\lambda}})
\end{aligned} \tag{5.13}
$$

We see $A_1 < A_2$ and $f_1 > f_2$. As such we can interpret the asyptotics to be an overlay of a slowly oscillating part with big amplitude and a quickly oscillating part with low amplitude. Because $A_1 < A_2$ we see from (5.13) that $\det M$ takes both negative and positive values and hence it must have zeros by the intermediate value theorem for continuous functions. This completes the proof of Lemma 4.8 because now we know that for any $q > 0$ there exists some $\lambda > 0$ such that (5.3) - (5.5) has a solution of the form (5.10).

5.2 An example

For the case where D is the unit disc and we have constant $p = 0$ and $q = 0.5$ we examine the plot of $\det M$ from the previous section. Equation (5.13) motivates us to regard $\det M$ not as a function of λ, but instead as a function of $\sqrt{\lambda}$.

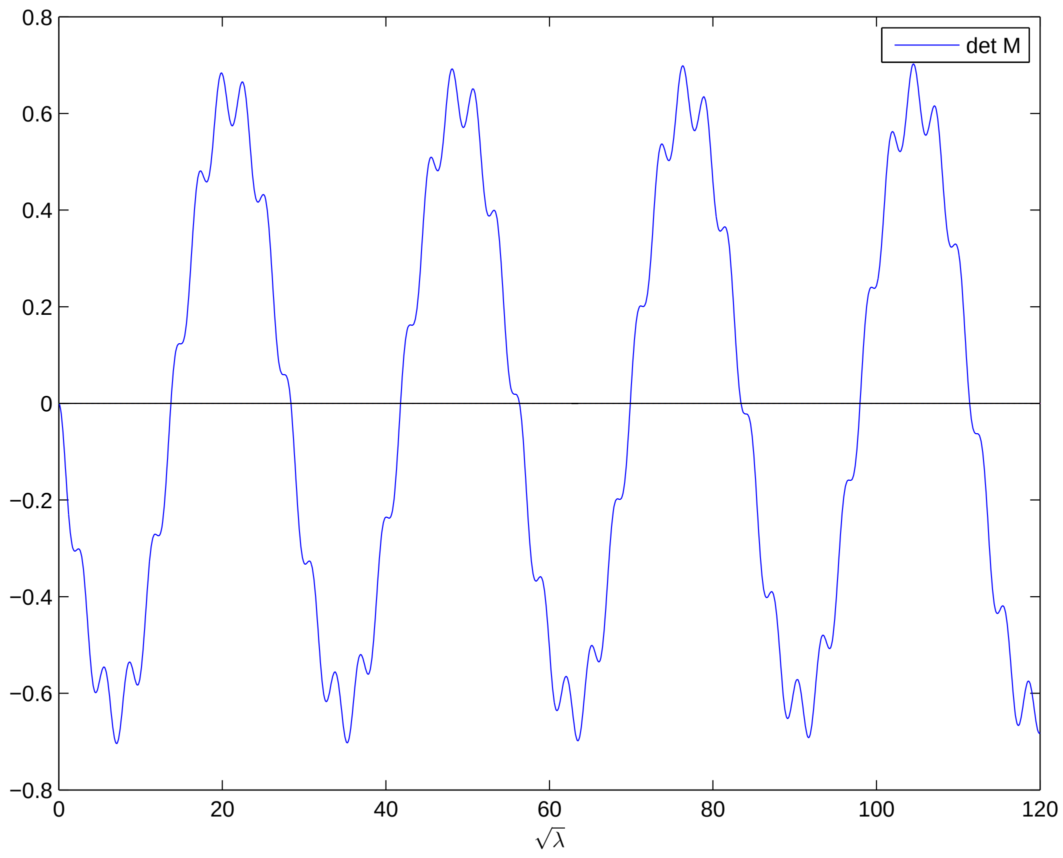

Figure 5.2: $\det M$ as a function of $\sqrt{\lambda}$ while $a = 1$ and $q = 0.5$

Our earlier deduction about how it looks like is confirmed. In fact we observe that the asymptotics are already clearly visible for 'small' λ.

Still being in the case where D is the unit circle and $p = 0$, the following table presents the first 6 positive zeros of $\det M$ for three different values of q. These are the first 6 interior transmission eigenvalues with eigenfunctions of the form (5.10).

	$q = 0.2$		$q = 0.5$		$q = 1$	
	$\sqrt{\lambda}$	λ	$\sqrt{\lambda}$	λ	$\sqrt{\lambda}$	λ
x_1	32.59	1062	13.71	187.9	7.375	54.39
x_2	66.23	4387	28.38	805.6	15.58	242.6
x_3	98.48	9698	41.80	1747	22.76	518.2
x_4	132.1	17450	56.34	3174	29.92	895.1
x_5	164.4	27020	69.90	4886	38.15	1455
x_6	198.0	39190	83.44	6961	45.22	2044

Table 5.1: Zeros of $\det M$, $p = 0$ and $a = 1$

5.3 The complex case

So far we have only considered the case where all variables defined in the introduction are real. What if we allow complex valued eigenfunctions u and w ? Then our existence result Theorem 4.7 clearly stays valid as we can just regard our real eigenfuctions as complex ones with zero imaginary part. Similarly we can see that our countability result Theorem 3.8 is still valid. If we have complex u and w in Definition 1.2, then both their real and imaginary parts also fulfill every demand except possibly the non-triviality. However they can not both be trivial because then u, w would also be trivial. So there can not be an interior transmission eigenvalue with complex eigenfunctions u, w that does not also have real eigenfunctions and Theorem 3.8 stays valid.

We could also allow for a complex index of refraction. In this case our analysis does not help us, but [4, Theorem 8.12] does state that there exist no interior transmission eigenvalues if the index of refraction has imaginary part unequal to zero. Note that [4] only accepts strong solutions and $p = 0$ though. For a definite answer to the question whether the theorem still holds true in our setting the reader has to be referred to the name of the chapter.

Finally demanding simply $\lambda \in \mathbb{C}$ instead of $\lambda > 0$ is also an option. We will examine this in the case of the example from the previous section, i.e. $p = 0$, $q = \frac{1}{2}$, D is the unit circle and we only search for eigenfunctions of the form (5.10). First of all J_0 is an even function as one can see from (5.9). Hence equation (5.13) is also valid for $\lambda \to -\infty$ and we expect a similar behaviour of it's zeros on the negative real axis as on the positive real axis. $\lambda = 0$ is clearly a zero of $\det M$, but is an uninteresting case anyway. So what of $\lambda \in \mathbb{C}\backslash\mathbb{R}$? If we find a nonreal zero of

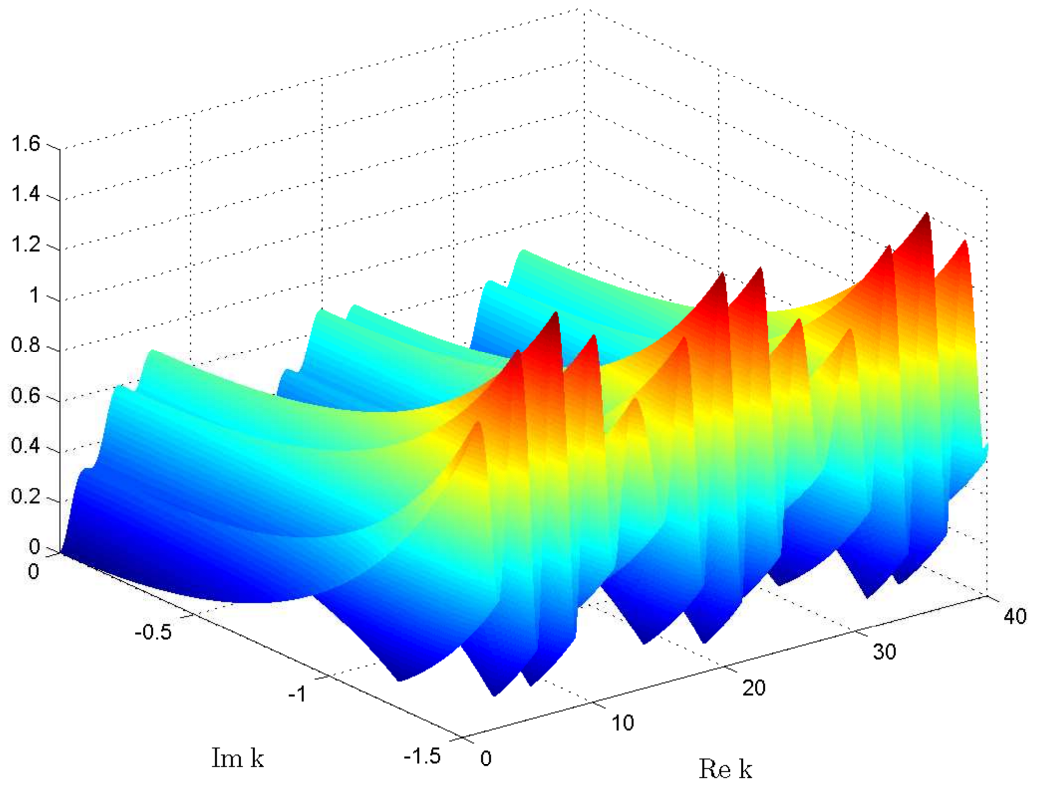

Figure 5.3: $|\det M|$ as a function of k while $a = 1$ and $q = 0.5$

$\det M$, we can proceed as we did in the real case. Without going into the details how (5.10) defines complex solutions, at this point we simply examine $\det M$. We look at (5.10) as well as (5.11) and see that we can formally regard $\det M$ as a function of $k := \sqrt{\lambda}$. If k is not real, then neither is k^2 and the functions defined through (5.11) solve (5.3),(5.4) with k^2 instead of λ.

Figure 5.3 shows conspicuous behavior of $|\det M|$ in the area where the imaginary part of k is between -1 and -1.5. Because $\det M$ is a smooth function, the dents in $|\det M|$ near zero suggest that $\det M$ might indeed have a zero at those points. In Figure 5.4 we can see one of the critical areas magnified and in Figure 5.5 the same area with contour lines for $Im(\det M) = 0$ and $Re(\det M) = 0$ as data.

We can clearly see three intersections of the contour lines of Figure 5.5. At these points we have $Im(\det M) = Re(\det M) = 0$ and hence $\det M = 0$. So our suspicion that for these values of k we can define nontrivial functions satisfying our boundary conditions through (5.11) remains. An analytical proof will however not be given here.

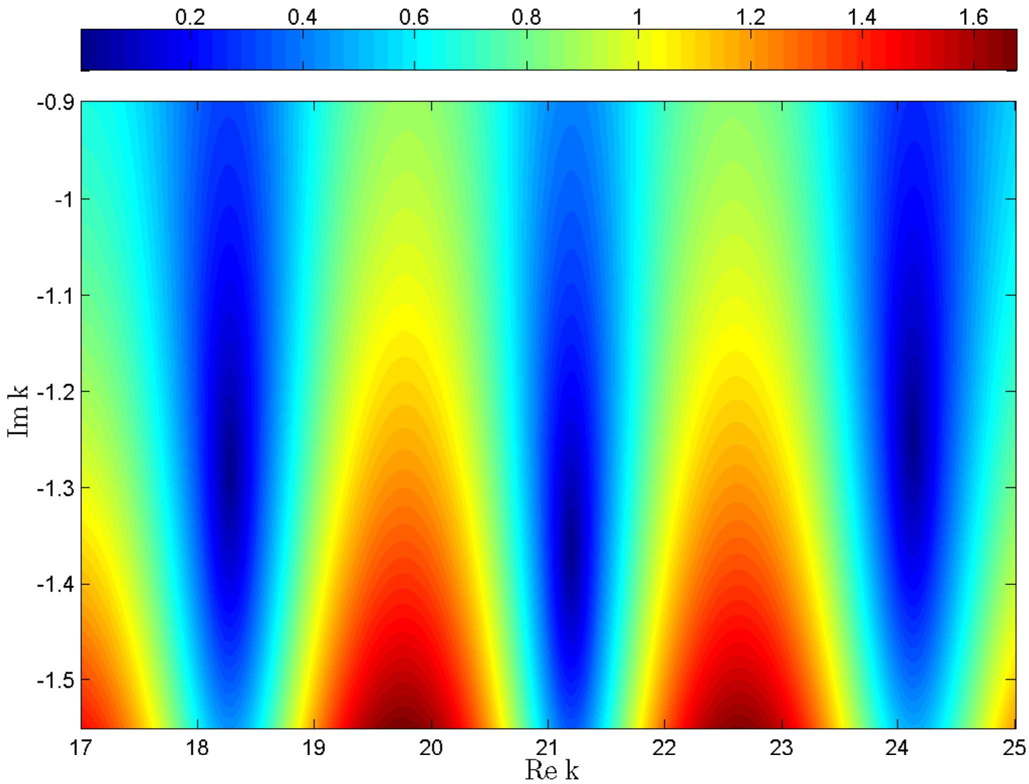

Figure 5.4: $|\det M|$ as a function of k while $a = 1$ and $q = 0.5$

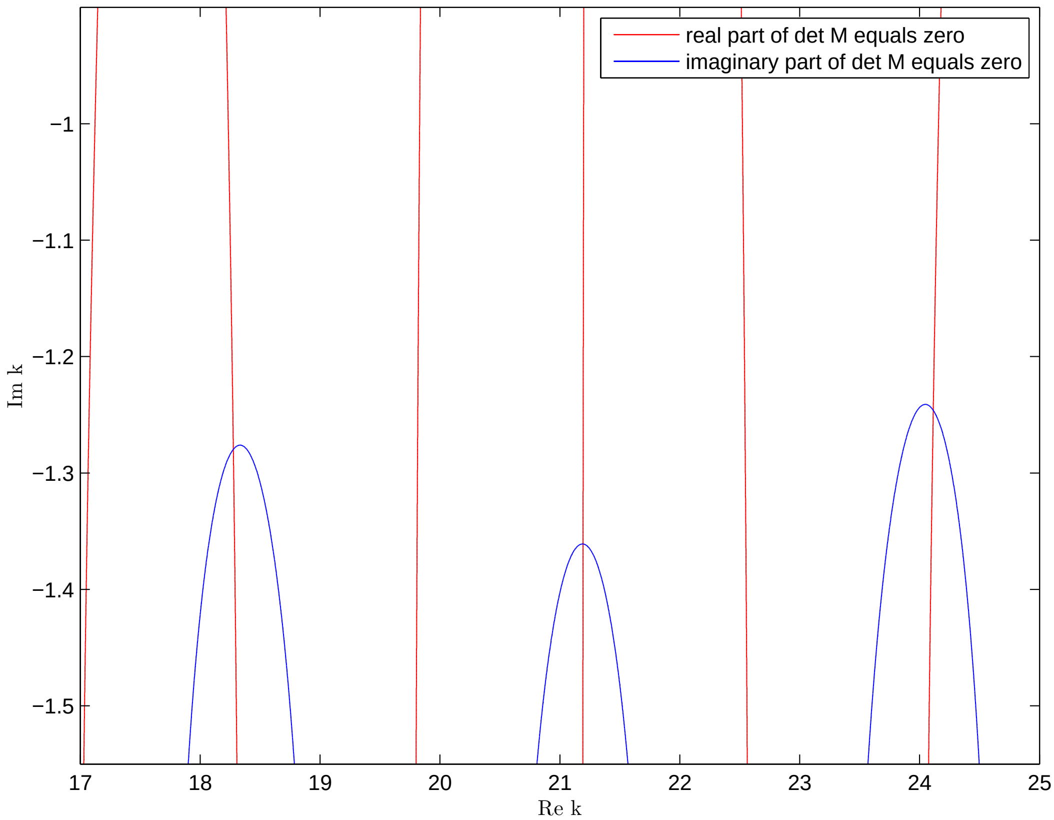

Figure 5.5: Contour lines $Re(\det M(k)) = 0$ and $Im(\det M(k)) = 0$

5.4 Opportunities for further studies

Regarding the distribution of interior transmission eigenvalues we have shown that at least one exists and the only possible accumulation point is infinity. Equation (5.13) does however suggest that for constant q and $p = 0$ there are in fact infinitely many of them. [5, Theorem 2] furthermore states that in $\mathbb{R}^3$ this is also true for spherically stratified q. A natural question in this context is whether this holds true for other sorts of refraction indices, too. Another interesting aspect to study would be how they are precisely distributed. [3] gives a result regarding upper and lower bounds for the smallest eigenvalue. Additionally one could strive to extend that to the kth eigenvalue or search for asymptotic behaviour of the distance between eigenvalues if there exist infinitely many.

On a more personal level the logical next step in my studies would be to better understand the relevance of interior transmission eigenvalues in applications. [4] states that their existence poses problems for the numerical implementation of reconstruction methods for unknown index of refraction. Studying further in this direction would then lead to the question how we can use our hard-earned results Theorem 3.8 and Theorem 4.7 to overcome these difficulties.

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Eidesstattliche Erklärung

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